Appendix C: Variation

Direct Variation

There are two basic types of linear models. The more general model has a y-intercept that is nonzero.

$$y = mx + b, \quad b \neq 0$$

The simpler model $y = kx$ has a y-intercept that is zero. In the simpler model, y is said to **vary directly** as x, or to be **directly proportional** to x.

Direct Variation

The following statements are equivalent.

1. y **varies directly** as x.
2. y is **directly proportional** to x.
3. $y = kx$ for some nonzero constant k.

k is the **constant of variation** or the **constant of proportionality.**

What you should learn

- Write mathematical models for direct variation.
- Write mathematical models for direct variation as an nth power.
- Write mathematical models for inverse variation.
- Write mathematical models for joint variation.

Why you should learn it

You can use functions as models to represent a wide variety of real-life data sets. For instance, in Exercise 61 on page C7, a variation model can be used to model the water temperatures of the ocean at various depths.

EXAMPLE 1 Direct Variation

In Pennsylvania, the state income tax is directly proportional to *gross income*. You are working in Pennsylvania and your state income tax deduction is \$46.05 for a gross monthly income of \$1500. Find a mathematical model that gives the Pennsylvania state income tax in terms of gross income.

Solution

Verbal Model: State income tax $=$ k $\cdot$ Gross income

Labels: State income tax $= y$ (dollars)
Gross income $= x$ (dollars)
Income tax rate $= k$ (percent in decimal form)

Equation: $y = kx$

To solve for k, substitute the given information in the equation $y = kx$, and then solve for k.

$y = kx$ Write direct variation model.

$46.05 = k(1500)$ Substitute $y = 46.05$ and $x = 1500$.

$0.0307 = k$ Simplify.

So, the equation (or model) for state income tax in Pennsylvania is

$$y = 0.0307x.$$

In other words, Pennsylvania has a state income tax rate of 3.07% of gross income. The graph of this equation is shown in Figure C.1.

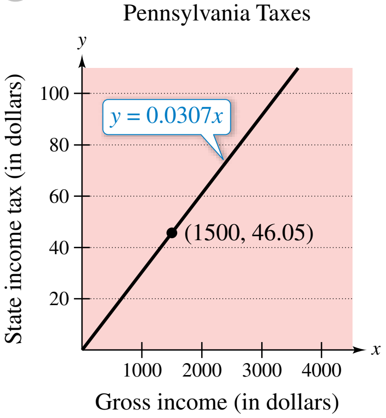


Figure C.1

Direct Variation as an *n*th Power

Another type of direct variation relates one variable to a *power* of another variable. For example, in the formula for the area of a circle

$$A = \pi r^2$$

the area A is directly proportional to the square of the radius r. Note that for this formula, π is the constant of proportionality.

Direct Variation as an *n*th Power

The following statements are equivalent.

1. y **varies directly as the** n**th power** of x.
2. y is **directly proportional to the** n**th power** of x.
3. $y = kx^n$ for some constant k.

Remark

Note that the direct variation model $y = kx$ is a special case of $y = kx^n$ with $n = 1$.

EXAMPLE 2 Direct Variation as an *n*th Power

The distance a ball rolls down an inclined plane is directly proportional to the square of the time it rolls. During the first second, the ball rolls 8 feet. (See Figure C.2.)

a. Write an equation relating the distance traveled to the time.

b. How far will the ball roll during the first 3 seconds?

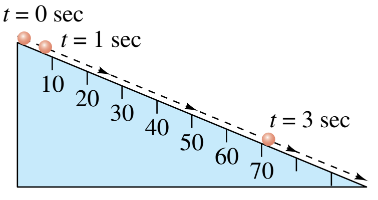


Figure C.2

Solution

a. Letting d be the distance (in feet) the ball rolls and letting t be the time (in seconds), you have

$$d = kt^2.$$

Now, because $d = 8$ when $t = 1$, you can see that $k = 8$, as follows.

$$d = kt^2$$

$$8 = k(1)^2$$

$$8 = k$$

So, the equation relating distance to time is

$$d = 8t^2.$$

b. When $t = 3$, the distance traveled is $d = 8(3)^2 = 8(9) = 72$ feet.

In Examples 1 and 2, the direct variations are such that an *increase* in one variable corresponds to an *increase* in the other variable. This is also true in the model $d = \frac{1}{5}F$, $F > 0$, where an increase in F results in an increase in d. You should not, however, assume that this always occurs with direct variation. For example, in the model $y = -3x$, an increase in x results in a *decrease* in y, and yet y is said to vary directly as x.

Inverse Variation

Inverse Variation

The following statements are equivalent.

1. y **varies inversely** as x.
2. y is **inversely proportional** to x.
3. $y = \dfrac{k}{x}$ for some constant k.

If x and y are related by an equation of the form $y = k/x^n$, then y varies inversely as the nth power of x (or y is inversely proportional to the nth power of x).

Some applications of variation involve problems with *both* direct and inverse variation in the same model. These types of models are said to have **combined variation.**

EXAMPLE 3 Direct and Inverse Variation

A gas law states that the volume of an enclosed gas varies directly as the temperature *and* inversely as the pressure, as shown in Figure C.3. The pressure of a gas is 0.75 kilogram per square centimeter when the temperature is 294 K and the volume is 8000 cubic centimeters. (a) Write an equation relating pressure, temperature, and volume. (b) Find the pressure when the temperature is 300 K and the volume is 7000 cubic centimeters.

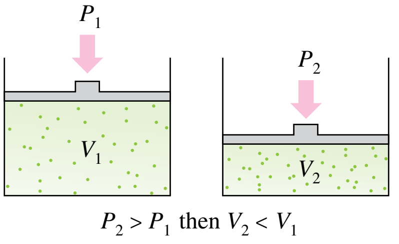


Figure C.3 ***If the temperature is held constant and pressure increases, then volume decreases.***

Solution

a. Let V be volume (in cubic centimeters), let P be pressure (in kilograms per square centimeter), and let T be temperature (in Kelvin). Because V varies directly as T and inversely as P, you have

$$V = \frac{kT}{P}.$$

Now, because $P = 0.75$ when $T = 294$ and $V = 8000$, you have

$$8000 = \frac{k(294)}{0.75}$$

$$k = \frac{6000}{294} = \frac{1000}{49}.$$

So, the equation relating pressure, temperature, and volume is

$$V = \frac{1000}{49}\left(\frac{T}{P}\right).$$

b. When $T = 300$ and $V = 7000$, the pressure is

$$V = \frac{1000}{49}\left(\frac{300}{7000}\right) = \frac{300}{343} \approx 0.87 \text{ kilogram per square centimeter.}$$

Joint Variation

In Example 3, note that when a direct variation and an inverse variation occur in the same statement, they are coupled with the word "and." To describe two different *direct* variations in the same statement, the word **jointly** is used.

Joint Variation

The following statements are equivalent.

1. z **varies jointly** as x and y.
2. z is **jointly proportional** to x and y.
3. $z = kxy$ for some constant k.

If x, y, and z are related by an equation of the form

$$z = kx^n y^m$$

then z varies jointly as the nth power of x and the mth power of y.

EXAMPLE 4 Joint Variation

The *simple* interest for a certain savings account is jointly proportional to the time and the principal. After one quarter (3 months), the interest on a principal of \$5000 is \$43.75.

a. Write an equation relating the interest, principal, and time.

b. Find the interest after three quarters.

Solution

a. Let $I =$ interest (in dollars), $P =$ principal (in dollars), and $t =$ time (in years). Because I is jointly proportional to P and t, you have

$$I = kPt.$$

For $I = 43.75$, $P = 5000$, and $t = \frac{1}{4}$, you have

$$43.75 = k(5000)\left(\frac{1}{4}\right)$$

which implies that $k = 4(43.75)/5000 = 0.035$. So, the equation relating interest, principal, and time is

$$I = 0.035Pt$$

which is the familiar equation for simple interest where the constant of proportionality, 0.035, represents an annual interest rate of 3.5%.

b. When $P = \$5000$ and $t = \frac{3}{4}$, the interest is

$$I = (0.035)(5000)\left(\frac{3}{4}\right)$$

$$= \$131.25.$$

C Exercises

For instructions on how to use a graphing utility, see Appendix A.

In Exercises 1–6, fill in the blank(s).

1. Direct variation models can be described as "y varies directly as x," or "y is ________ to x."

2. In direct variation models of the form $y = kx$, k is called the ________ of ________.

3. The direct variation model $y = kx^n$ can be described as "y varies directly as the nth power of x," or "y is ________ to the nth power of x."

4. The mathematical model $y = \frac{k}{x}$ is an example of ________ variation.

5. Mathematical models that involve both direct and inverse variation are said to have ________ variation.

6. The joint variation model $z = kxy$ can be described as "z varies jointly as x and y," or "z is ________ to x and y."

Procedures and Problem Solving

Direct Variation **In Exercises 7–10, assume that y is directly proportional to x. Use the given x-value and y-value to find a linear model that relates y and x.**

7. $x = 5, \quad y = 12$

8. $x = 2, \quad y = 14$

9. $x = 10, \quad y = 2050$

10. $x = 6, \quad y = 580$

11. **Converting Units** On a yardstick with scales in inches and centimeters, you notice that 13 inches is approximately the same length as 33 centimeters. Use this information to find a mathematical model that relates centimeters to inches. Then use the model to find the numbers of centimeters in 10 inches and 20 inches.

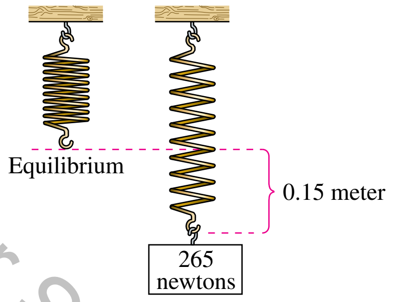
12. **Converting Units** When buying gasoline, you notice that 14 gallons of gasoline is approximately the same amount of gasoline as 53 liters. Use this information to find a linear model that relates gallons to liters. Then use the model to find the numbers of liters in 5 gallons and 25 gallons.

13. **Accounting** Property tax is based on the assessed value of a property. A house that has an assessed value of \$150,000 has a property tax of \$5520. Find a mathematical model that gives the amount of property tax in terms of the assessed value x of the property. Use the model to find the property tax on a house that has an assessed value of \$200,000.

14. **Accounting** State sales tax is based on retail price. An item that sells for \$145.99 has a sales tax of \$10.22. Find a mathematical model that gives the amount of sales tax in terms of the retail price x. Use the model to find the sales tax on a \$540.50 purchase.

Hooke's Law **In Exercises 15 and 16, use Hooke's Law for springs, which states that the distance a spring is stretched (or compressed) varies directly as the force on the spring.**

15. A force of 265 newtons stretches a spring 0.15 meter (see figure).



(a) How far will a force of 90 newtons stretch the spring?

(b) What force is required to stretch the spring 0.1 meter?

16. A force of 220 newtons stretches a spring 0.12 meter. What force is required to stretch the spring 0.16 meter?

Direct Variation as an *n*th Power **In Exercises 17–20, use the given value of k to complete the table for the direct variation model $y = kx^2$. Plot the points on a rectangular coordinate system.**

x	2	4	6	8	10
$y = kx^2$					

17. $k = 1$

18. $k = 2$

19. $k = \frac{1}{2}$

20. $k = \frac{1}{4}$

Ecology **In Exercises 21 and 22, use the fact that the diameter of the largest particle that can be moved by a stream varies approximately directly as the square of the velocity of the stream.**

21. A stream with a velocity of $\frac{1}{4}$ mile per hour can move coarse sand particles about 0.02 inch in diameter. Approximate the velocity required to carry particles 0.12 inch in diameter.

22. A stream of velocity v can move particles of diameter d or less. By what factor does d increase when the velocity is doubled?

Inverse Variation **In Exercises 23–26, use the given value of k to complete the table for the inverse variation model $y = k/x^2$. Plot the points on a rectangular coordinate system.**

x	2	4	6	8	10
$y = \frac{k}{x^2}$					

23. $k = 2$

24. $k = 5$

25. $k = 10$

26. $k = 20$

Identifying Direct or Inverse Variation **In Exercises 27–30, determine whether the variation model is of the form $y = kx$ or $y = k/x$ and find k.**

27.

x	5	10	15	20	25
y	1	$\frac{1}{2}$	$\frac{1}{3}$	$\frac{1}{4}$	$\frac{1}{5}$

28.

x	5	10	15	20	25
y	2	4	6	8	10

29.

x	5	10	15	20	25
y	-3.5	-7	-10.5	-14	-17.5

30.

x	5	10	15	20	25
y	24	12	8	6	$\frac{24}{5}$

Writing a Variation Model **In Exercises 31–40, find a mathematical model for the verbal statement.**

31. A varies directly as the square of r.

32. V varies directly as the cube of e.

33. y varies inversely as the square of x.

34. h varies inversely as the square root of s.

35. F varies directly as g and inversely as r^2.

36. z is jointly proportional to the square of x and the cube of y.

37. **Boyle's Law:** For a constant temperature, the pressure P of a gas is inversely proportional to the volume V of the gas.

38. **Logistic Growth:** The rate of growth R of a population is jointly proportional to the size S of the population and the difference between S and the maximum population size L that the environment can support.

39. **Newton's Law of Cooling:** The rate of change R of the temperature of an object is proportional to the difference between the temperature T of the object and the temperature T_e of the environment in which the object is placed.

40. **Newton's Law of Universal Gravitation:** The gravitational attraction F between two objects of masses m_1 and m_2 is proportional to the product of the masses and inversely proportional to the square of the distance r between the objects.

Describing the Variation in a Formula **In Exercises 41–46, write a sentence using the variation terminology of this section to describe the formula.**

41. ***Area of a triangle:*** $A = \frac{1}{2}bh$

42. ***Surface area of a sphere:*** $S = 4\pi r^2$

43. ***Volume of a sphere:*** $V = \frac{4}{3}\pi r^3$

44. ***Volume of a right circular cylinder:*** $V = \pi r^2 h$

45. ***Average speed:*** $r = \frac{d}{t}$

46. ***Free vibrations:*** $\omega = \sqrt{\frac{kg}{W}}$

Writing a Variation Model **In Exercises 47–54, find a mathematical model representing the statement. (In each case, determine the constant of proportionality.)**

47. A varies directly as r^2. ($A = 9\pi$ when $r = 3$.)

48. y varies inversely as x. ($y = 3$ when $x = 25$.)

49. y is inversely proportional to x. ($y = 7$ when $x = 4$.)

50. z varies jointly as x and y. ($z = 64$ when $x = 4$ and $y = 8$.)

51. F is jointly proportional to r and the third power of s. ($F = 4158$ when $r = 11$ and $s = 3$.)

52. P varies directly as x and inversely as the square of y. $\left(P = \frac{28}{3}\right.$ when $x = 42$ and $\left.y = 9.\right)$

53. z varies directly as the square of x and inversely as y. ($z = 6$ when $x = 6$ and $y = 4$.)

54. v varies jointly as p and q and inversely as the square of s. ($v = 1.5$ when $p = 4.1$, $q = 6.3$, and $s = 1.2$.)

Electrical Engineering **In Exercises 55 and 56, use the fact that the resistance of a wire carrying an electrical current is directly proportional to its length and inversely proportional to its cross-sectional area.**

55. In a 28-gauge copper wire that has a diameter of 0.0126 inch, the resistance is 66.17 ohms per thousand feet. What length of the wire has a resistance of 33.5 ohms?

56. A 14-foot piece of copper wire produces a resistance of 0.05 ohm. Use the constant of proportionality from Exercise 55 to find the diameter of the wire.

57. **Joint Variation** The work W (in joules) done when an object is lifted varies jointly with the mass m (in kilograms) of the object and the height h (in meters) that the object is lifted. The work done when a 120-kilogram object is lifted 1.8 meters is 2116.8 joules. How much work is done when a 100-kilogram object is lifted 1.5 meters?

58. **Restaurant Management** The prices of three sizes of pizza at a pizza shop are as follows.

9-inch: \$8.78, 12-inch: \$11.78, 15-inch: \$14.18

You might expect the price of a pizza to be directly proportional to its surface area. Is that the case for this pizza shop? If not, which size of pizza is the best buy?

59. **Fluid Mechanics** The velocity v of a fluid flowing in a conduit is inversely proportional to the cross-sectional area of the conduit. (Assume that the volume of the flow per unit of time is held constant.) Determine the change in the velocity of water flowing from a hose when a person places a finger over the end of the hose to decrease its cross-sectional area by 25%.

60. **Structural Engineering** The maximum load that can be safely supported by a horizontal beam varies jointly as the width of the beam and the square of its depth, and inversely as the length of the beam. Determine the changes in the maximum safe load under the following conditions.

(a) The width and length of the beam are doubled.

(b) The width and depth of the beam are doubled.

(c) All three of the dimensions are doubled.

(d) The depth of the beam is halved.

61. ***Why you should learn it*** *(p. C1)* An oceanographer took readings of the water temperatures C (in degrees Celsius) at several depths d (in meters). The data collected is shown in the table.

Depth, d	Temperature, C
1000	4.2°
2000	1.9°
3000	1.4°
4000	1.2°
5000	0.9°

(a) Sketch a scatter plot of the data.

(b) Does it appear that the data can be modeled by the inverse variation model $C = k/d$? If so, find k for each pair of coordinates.

(c) Determine the mean value of k from part (b) to find the inverse variation model $C = k/d$.

(d) Use a graphing utility to plot the data points and the inverse model in part (c).

(e) Use the model to approximate the depth at which the water temperature is 3°C.

62. **Physics** An experiment in a physics lab requires a student to measure the compressed lengths y (in centimeters) of a spring when various forces of F pounds are applied. The data are shown in the table.

DATA

Spreadsheet at LarsonPrecalculus.com

Force, F	Length, y
0	0
2	1.15
4	2.3
6	3.45
8	4.6
10	5.75
12	6.9

(a) Sketch a scatter plot of the data.

(b) Does it appear that the data can be modeled by Hooke's Law? If so, estimate k. (See Exercises 15 and 16.)

(c) Use the model in part (b) to approximate the force required to compress the spring 9 centimeters.

Conclusions

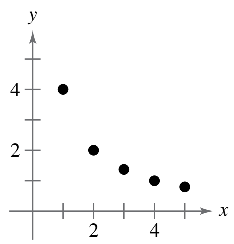
True or False? **In Exercises 63 and 64, decide whether the statement is true or false. Justify your answer.**

63. If y varies directly as x, then when x increases, y will increase as well.

64. In the equation for kinetic energy, $E = \frac{1}{2}mv^2$, the amount of kinetic energy E is directly proportional to the mass m of an object and the square of its velocity v.

Think About It **In Exercises 65 and 66, use the graph to determine whether y varies directly as some power of x or inversely as some power of x. Explain.**

65.



66.