A.1 Real Numbers and Their Properties

Real numbers can represent many real-life quantities. For example, in Exercises 55–58 on page A12, you will use real numbers to represent the federal deficit.

- Represent and classify real numbers.
- Order real numbers and use inequalities.
- Find the absolute values of real numbers and find the distance between two real numbers.
- Evaluate algebraic expressions.
- Use the basic rules and properties of algebra.

Real Numbers

Real numbers can describe quantities in everyday life such as age, miles per gallon, and population. Symbols such as

$$-5,\ 9,\ 0,\ \tfrac{4}{3},\ 0.666\ .\ .\ .\ ,\ 28.21,\ \sqrt{2},\ \pi,\ \text{and}\ \sqrt[3]{-32}$$

represent real numbers. Here are some important **subsets** (each member of a subset B is also a member of a set A) of the real numbers. The three dots, called *ellipsis points,* indicate that the pattern continues indefinitely.

$\{1, 2, 3, 4, .\ .\ .\}$ Set of natural numbers

$\{0, 1, 2, 3, 4, .\ .\ .\}$ Set of whole numbers

$\{.\ .\ .\ , -3, -2, -1, 0, 1, 2, 3, .\ .\ .\}$ Set of integers

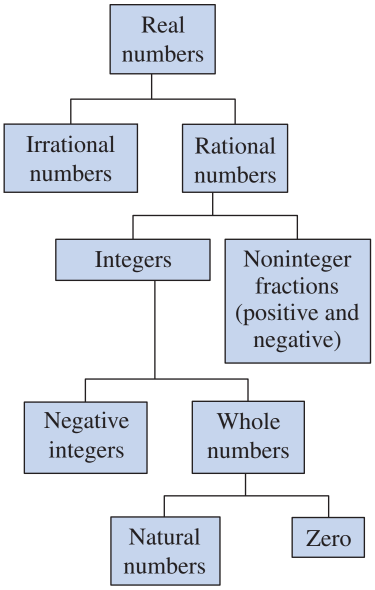
A real number is **rational** when it can be written as the ratio p/q of two integers, where $q \neq 0$. For instance, the numbers

$$\tfrac{1}{3} = 0.3333\ .\ .\ . = 0.\overline{3},\ \tfrac{1}{8} = 0.125,\ \text{and}\ \tfrac{125}{111} = 1.126126\ .\ .\ . = 1.\overline{126}$$

are rational. The decimal representation of a rational number either repeats $\left(\text{as in } \tfrac{173}{55} = 3.1\overline{45}\right)$ or terminates $\left(\text{as in } \tfrac{1}{2} = 0.5\right)$. A real number that cannot be written as the ratio of two integers is called **irrational.** Irrational numbers have infinite nonrepeating decimal representations. For instance, the numbers

$$\sqrt{2} = 1.4142135\ .\ .\ . \approx 1.41 \quad \text{and} \quad \pi = 3.1415926\ .\ .\ . \approx 3.14$$

are irrational. (The symbol $\approx$ means "is approximately equal to.") Figure A.1 shows subsets of real numbers and their relationships to each other.



Subsets of real numbers

Figure A.1

EXAMPLE 1 Classifying Real Numbers

Determine which numbers in the set $\left\{-13, -\sqrt{5}, -1, -\tfrac{1}{3}, 0, \tfrac{5}{8}, \sqrt{2}, \pi, 7\right\}$ are (a) natural numbers, (b) whole numbers, (c) integers, (d) rational numbers, and (e) irrational numbers.

Solution

a. Natural numbers: $\{7\}$

b. Whole numbers: $\{0, 7\}$

c. Integers: $\{-13, -1, 0, 7\}$

d. Rational numbers: $\left\{-13, -1, -\dfrac{1}{3}, 0, \dfrac{5}{8}, 7\right\}$

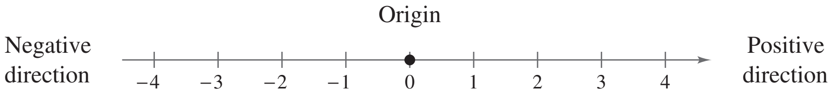
e. Irrational numbers: $\left\{-\sqrt{5}, \sqrt{2}, \pi\right\}$

✓ ***Checkpoint*** *Audio-video solution in English & Spanish at LarsonPrecalculus.com.*

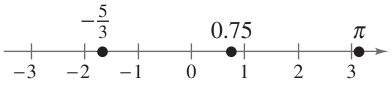
Repeat Example 1 for the set $\left\{-\pi, -\tfrac{1}{4}, \tfrac{6}{3}, \tfrac{1}{2}\sqrt{2}, -7.5, -1, 8, -22\right\}$.

Michael G Smith/Shutterstock.com

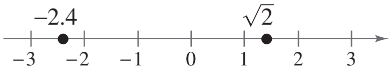
Real numbers are represented graphically on the **real number line.** When you draw a point on the real number line that corresponds to a real number, you are **plotting** the real number. The point 0 on the real number line is the **origin.** Numbers to the right of 0 are positive, and numbers to the left of 0 are negative, as shown below. The term **nonnegative** describes a number that is either positive or zero.



As illustrated below, there is a *one-to-one correspondence* between real numbers and points on the real number line.



Every real number corresponds to exactly one point on the real number line.



Every point on the real number line corresponds to exactly one real number.

EXAMPLE 2 Plotting Points on the Real Number Line

Plot the real numbers on the real number line.

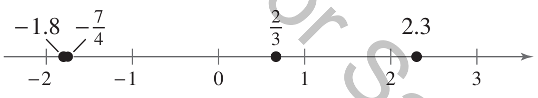
a. $-\frac{7}{4}$

b. 2.3

c. $\frac{2}{3}$

d. -1.8

Solution The following figure shows all four points.



a. The point representing the real number $-\frac{7}{4} = -1.75$ lies between -2 and -1, but closer to -2, on the real number line.

b. The point representing the real number 2.3 lies between 2 and 3, but closer to 2, on the real number line.

c. The point representing the real number $\frac{2}{3} = 0.666 \ldots$ lies between 0 and 1, but closer to 1, on the real number line.

d. The point representing the real number -1.8 lies between -2 and -1, but closer to -2, on the real number line. Note that the point representing -1.8 lies slightly to the left of the point representing $-\frac{7}{4}$.

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Plot the real numbers on the real number line.

a. $\frac{5}{2}$ **b.** -1.6

c. $-\frac{3}{4}$ **d.** 0.7

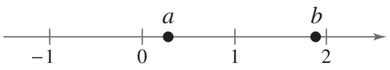
Ordering Real Numbers

One important property of real numbers is that they are *ordered.*

Definition of Order on the Real Number Line

If a and b are real numbers, then a is less than b when $b - a$ is positive. The **inequality** $a < b$ denotes the **order** of a and b. This relationship can also be described by saying that b is *greater than* a and writing $b > a$. The inequality $a \leq b$ means that a is *less than or equal to* b, and the inequality $b \geq a$ means that b is *greater than or equal to* a. The symbols $<$, $>$, $\leq$, and $\geq$ are *inequality symbols.*

Geometrically, this definition implies that $a < b$ if and only if a lies to the *left* of b on the real number line, as shown in Figure A.2.



$a < b$ if and only if a lies to the left of b.

Figure A.2

EXAMPLE 3 Ordering Real Numbers

Place the appropriate inequality symbol ($<$ or $>$) between the pair of real numbers.

a. $-3, 0$ **b.** $-2, -4$ **c.** $\frac{1}{4}, \frac{1}{3}$ **d.** $-\frac{1}{5}, -\frac{1}{2}$

Solution

a. Because -3 lies to the left of 0 on the real number line, as shown in Figure A.3, you can say that -3 is *less than* 0, and write $-3 < 0$.

b. Because -2 lies to the right of -4 on the real number line, as shown in Figure A.4, you can say that -2 is *greater than* -4, and write $-2 > -4$.

c. Because $\frac{1}{4}$ lies to the left of $\frac{1}{3}$ on the real number line, as shown in Figure A.5, you can say that $\frac{1}{4}$ is *less than* $\frac{1}{3}$, and write $\frac{1}{4} < \frac{1}{3}$.

d. Because $-\frac{1}{5}$ lies to the right of $-\frac{1}{2}$ on the real number line, as shown in Figure A.6, you can say that $-\frac{1}{5}$ is *greater than* $-\frac{1}{2}$, and write $-\frac{1}{5} > -\frac{1}{2}$.

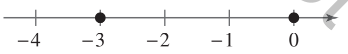


Figure A.3

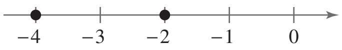


Figure A.4

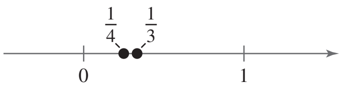


Figure A.5

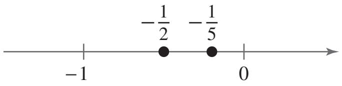


Figure A.6

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Place the appropriate inequality symbol ($<$ or $>$) between the pair of real numbers.

a. $1, -5$ **b.** $\frac{3}{2}, 7$ **c.** $-\frac{2}{3}, -\frac{3}{4}$ **d.** $-3.5, 1$

EXAMPLE 4 Interpreting Inequalities

Describe the subset of real numbers that the inequality represents.

a. $x \leq 2$ **b.** $-2 \leq x < 3$

Solution

a. The inequality $x \leq 2$ denotes all real numbers less than or equal to 2, as shown in Figure A.7.

b. The inequality $-2 \leq x < 3$ means that $x \geq -2$ *and* $x < 3$. This "double inequality" denotes all real numbers between -2 and 3, including -2 but not including 3, as shown in Figure A.8.

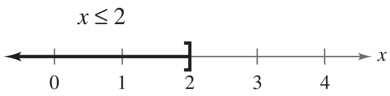

Figure A.7

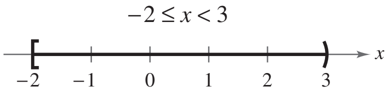

Figure A.8

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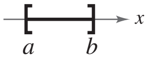
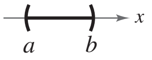
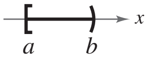
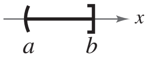
Describe the subset of real numbers that the inequality represents.

a. $x > -3$ **b.** $0 < x \leq 4$

Inequalities can describe subsets of real numbers called **intervals.** In the bounded intervals below, the real numbers a and b are the **endpoints** of each interval. The endpoints of a closed interval are included in the interval, whereas the endpoints of an open interval are not included in the interval.

REMARK The reason that the four types of intervals at the right are called *bounded* is that each has a finite length. An interval that does not have a finite length is *unbounded* (see below).

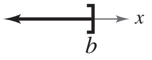
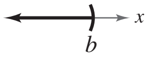
Bounded Intervals on the Real Number Line

Notation	Interval Type	Inequality	Graph
$[a, b]$	Closed	$a \leq x \leq b$	
(a, b)	Open	$a < x < b$	
$[a, b)$		$a \leq x < b$	
$(a, b]$		$a < x \leq b$	

The symbols ∞, **positive infinity,** and $-\infty$, **negative infinity,** do not represent real numbers. They are simply convenient symbols used to describe the unboundedness of an interval such as $(1, \infty)$ or $(-\infty, 3]$.

REMARK Whenever you write an interval containing ∞ or $-\infty$, always use a parenthesis and never a bracket next to these symbols. This is because ∞ and $-\infty$ are never an endpoint of an interval and therefore are not included in the interval.

Unbounded Intervals on the Real Number Line

Notation	Interval Type	Inequality	Graph
$[a, \infty)$		$x \geq a$	
(a, ∞)	Open	$x > a$	
$(-\infty, b]$		$x \leq b$	
$(-\infty, b)$	Open	$x < b$	
$(-\infty, \infty)$	Entire real line	$-\infty < x < \infty$	

EXAMPLE 5 Interpreting Intervals

a. The interval $(-1, 0)$ consists of all real numbers greater than -1 and less than 0.

b. The interval $[2, \infty)$ consists of all real numbers greater than or equal to 2.

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Give a verbal description of the interval $[-2, 5)$.

EXAMPLE 6 Using Inequalities to Represent Intervals

a. The inequality $c \leq 2$ can represent the statement "c is at most 2."

b. The inequality $-3 < x \leq 5$ can represent "all x in the interval $(-3, 5]$."

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Use inequality notation to represent the statement "x is greater than -2 and at most 4."

Absolute Value and Distance

The **absolute value** of a real number is its *magnitude,* or the distance between the origin and the point representing the real number on the real number line.

Definition of Absolute Value

If a is a real number, then the absolute value of a is

$$|a| = \begin{cases} a, & \text{if } a \geq 0 \\ -a, & \text{if } a < 0 \end{cases}.$$

Notice in this definition that the absolute value of a real number is never negative. For instance, if $a = -5$, then $|-5| = -(-5) = 5$. The absolute value of a real number is either positive or zero. Moreover, 0 is the only real number whose absolute value is 0. So, $|0| = 0$.

EXAMPLE 7 Finding Absolute Values

a. $|-15| = 15$ **b.** $\left|\dfrac{2}{3}\right| = \dfrac{2}{3}$

c. $|-4.3| = 4.3$ **d.** $-|-6| = -(6) = -6$

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Evaluate each expression.

a. $|1|$ **b.** $-\left|\dfrac{3}{4}\right|$

c. $\dfrac{2}{|-3|}$ **d.** $-|0.7|$

EXAMPLE 8 Evaluating the Absolute Value of a Number

Evaluate $\dfrac{|x|}{x}$ for (a) $x > 0$ and (b) $x < 0$.

Solution

a. If $x > 0$, then $|x| = x$ and $\dfrac{|x|}{x} = \dfrac{x}{x} = 1$.

b. If $x < 0$, then $|x| = -x$ and $\dfrac{|x|}{x} = \dfrac{-x}{x} = -1$.

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Evaluate $\dfrac{|x + 3|}{x + 3}$ for (a) $x > -3$ and (b) $x < -3$.

The **Law of Trichotomy** states that for any two real numbers a and b, *precisely* one of three relationships is possible:

$a = b, \quad a < b, \quad \text{or} \quad a > b.$ Law of Trichotomy

EXAMPLE 9 Comparing Real Numbers

Place the appropriate symbol ($<$, $>$, or $=$) between the pair of real numbers.

a. $|-4| \;\square\; |3|$ **b.** $|-10| \;\square\; |10|$ **c.** $-|-7| \;\square\; |-7|$

Solution

a. $|-4| > |3|$ because $|-4| = 4$ and $|3| = 3$, and 4 is greater than 3.

b. $|-10| = |10|$ because $|-10| = 10$ and $|10| = 10$.

c. $-|-7| < |-7|$ because $-|-7| = -7$ and $|-7| = 7$, and -7 is less than 7.

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Place the appropriate symbol ($<$, $>$, or $=$) between the pair of real numbers.

a. $|-3| \;\square\; |4|$ **b.** $-|-4| \;\square\; -|4|$ **c.** $|-3| \;\square\; -|-3|$

Properties of Absolute Values

1. $|a| \geq 0$

2. $|-a| = |a|$

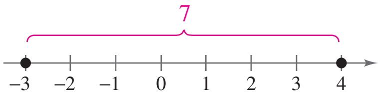
3. $|ab| = |a||b|$

4. $\left|\dfrac{a}{b}\right| = \dfrac{|a|}{|b|}, \quad b \neq 0$

Absolute value can be used to define the distance between two points on the real number line. For instance, the distance between -3 and 4 is

$$\begin{aligned}|-3 - 4| &= |-7| \\ &= 7\end{aligned}$$

as shown in Figure A.9.



The distance between -3 and 4 is 7.
Figure A.9

Distance Between Two Points on the Real Number Line

Let a and b be real numbers. The **distance between *a* and *b*** is

$$d(a, b) = |b - a| = |a - b|.$$

EXAMPLE 10 Finding a Distance

Find the distance between -25 and 13.

Solution

The distance between -25 and 13 is

$$|-25 - 13| = |-38| = 38. \qquad \text{Distance between } -25 \text{ and } 13$$

The distance can also be found as follows.

$$|13 - (-25)| = |38| = 38 \qquad \text{Distance between } -25 \text{ and } 13$$

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a. Find the distance between 35 and -23.

b. Find the distance between -35 and -23.

c. Find the distance between 35 and 23.

Algebraic Expressions

One characteristic of algebra is the use of letters to represent numbers. The letters are **variables,** and combinations of letters and numbers are **algebraic expressions.** Here are a few examples of algebraic expressions.

$$5x, \quad 2x - 3, \quad \frac{4}{x^2 + 2}, \quad 7x + y$$

Definition of an Algebraic Expression

An **algebraic expression** is a collection of letters **(variables)** and real numbers **(constants)** combined using the operations of addition, subtraction, multiplication, division, and exponentiation.

The **terms** of an algebraic expression are those parts that are separated by *addition.* For example, $x^2 - 5x + 8 = x^2 + (-5x) + 8$ has three terms: x^2 and $-5x$ are the **variable terms** and 8 is the **constant term.** The numerical factor of a term is called the **coefficient.** For instance, the coefficient of $-5x$ is -5, and the coefficient of x^2 is 1.

EXAMPLE 11 Identifying Terms and Coefficients

	Algebraic Expression	Terms	Coefficients
a.	$5x - \frac{1}{7}$	$5x, -\frac{1}{7}$	$5, -\frac{1}{7}$
b.	$2x^2 - 6x + 9$	$2x^2, -6x, 9$	$2, -6, 9$
c.	$\frac{3}{x} + \frac{1}{2}x^4 - y$	$\frac{3}{x}, \frac{1}{2}x^4, -y$	$3, \frac{1}{2}, -1$

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Identify the terms and coefficients of $-2x + 4$.

To **evaluate** an algebraic expression, substitute numerical values for each of the variables in the expression, as shown in the next example.

EXAMPLE 12 Evaluating Algebraic Expressions

	Expression	Value of Variable	Substitute.	Value of Expression
a.	$-3x + 5$	$x = 3$	$-3(3) + 5$	$-9 + 5 = -4$
b.	$3x^2 + 2x - 1$	$x = -1$	$3(-1)^2 + 2(-1) - 1$	$3 - 2 - 1 = 0$
c.	$\frac{2x}{x + 1}$	$x = -3$	$\frac{2(-3)}{-3 + 1}$	$\frac{-6}{-2} = 3$

Note that you must substitute the value for *each* occurrence of the variable.

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Evaluate $4x - 5$ when $x = 0$.

Use the **Substitution Principle** to evaluate algebraic expressions. It states that "If $a = b$, then b can replace a in any expression involving a." In Example 12(a), for instance, 3 is *substituted* for x in the expression $-3x + 5$.

Basic Rules of Algebra

There are four arithmetic operations with real numbers: *addition, multiplication, subtraction,* and *division,* denoted by the symbols $+$, $\times$ or $\cdot$, $-$, and $\div$ or $/$, respectively. Of these, addition and multiplication are the two primary operations. Subtraction and division are the inverse operations of addition and multiplication, respectively.

Definitions of Subtraction and Division

Subtraction: Add the opposite. **Division:** Multiply by the reciprocal.

$$a - b = a + (-b) \qquad \text{If } b \neq 0, \text{ then } a/b = a\left(\frac{1}{b}\right) = \frac{a}{b}.$$

In these definitions, $-b$ is the **additive inverse** (or opposite) of b, and $1/b$ is the **multiplicative inverse** (or reciprocal) of b. In the fractional form a/b, a is the **numerator** of the fraction and b is the **denominator.**

Because the properties of real numbers below are true for variables and algebraic expressions as well as for real numbers, they are often called the **Basic Rules of Algebra.** Try to formulate a verbal description of each property. For instance, the first property states that *the order in which two real numbers are added does not affect their sum.*

Basic Rules of Algebra

Let a, b, and c be real numbers, variables, or algebraic expressions.

Property		**Example**
Commutative Property of Addition:	$a + b = b + a$	$4x + x^2 = x^2 + 4x$
Commutative Property of Multiplication:	$ab = ba$	$(4 - x)x^2 = x^2(4 - x)$
Associative Property of Addition:	$(a + b) + c = a + (b + c)$	$(x + 5) + x^2 = x + (5 + x^2)$
Associative Property of Multiplication:	$(ab)c = a(bc)$	$(2x \cdot 3y)(8) = (2x)(3y \cdot 8)$
Distributive Properties:	$a(b + c) = ab + ac$	$3x(5 + 2x) = 3x \cdot 5 + 3x \cdot 2x$
	$(a + b)c = ac + bc$	$(y + 8)y = y \cdot y + 8 \cdot y$
Additive Identity Property:	$a + 0 = a$	$5y^2 + 0 = 5y^2$
Multiplicative Identity Property:	$a \cdot 1 = a$	$(4x^2)(1) = 4x^2$
Additive Inverse Property:	$a + (-a) = 0$	$5x^3 + (-5x^3) = 0$
Multiplicative Inverse Property:	$a \cdot \frac{1}{a} = 1, \quad a \neq 0$	$(x^2 + 4)\left(\frac{1}{x^2 + 4}\right) = 1$

Because subtraction is defined as "adding the opposite," the Distributive Properties are also true for subtraction. For instance, the "subtraction form" of $a(b + c) = ab + ac$ is $a(b - c) = ab - ac$. Note that the operations of subtraction and division are neither commutative nor associative. The examples

$$7 - 3 \neq 3 - 7 \quad \text{and} \quad 20 \div 4 \neq 4 \div 20$$

show that subtraction and division are not commutative. Similarly

$$5 - (3 - 2) \neq (5 - 3) - 2 \quad \text{and} \quad 16 \div (4 \div 2) \neq (16 \div 4) \div 2$$

demonstrate that subtraction and division are not associative.

EXAMPLE 13 Identifying Rules of Algebra

Identify the rule of algebra illustrated by the statement.

a. $(5x^3)2 = 2(5x^3)$

b. $(4x + 3) - (4x + 3) = 0$

c. $7x \cdot \dfrac{1}{7x} = 1, \quad x \neq 0$

d. $(2 + 5x^2) + x^2 = 2 + (5x^2 + x^2)$

Solution

a. This statement illustrates the Commutative Property of Multiplication. In other words, you obtain the same result whether you multiply $5x^3$ by 2, or 2 by $5x^3$.

b. This statement illustrates the Additive Inverse Property. In terms of subtraction, this property states that when any expression is subtracted from itself the result is 0.

c. This statement illustrates the Multiplicative Inverse Property. Note that x must be a nonzero number. The reciprocal of x is undefined when x is 0.

d. This statement illustrates the Associative Property of Addition. In other words, to form the sum $2 + 5x^2 + x^2$, it does not matter whether 2 and $5x^2$, or $5x^2$ and x^2 are added first.

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Identify the rule of algebra illustrated by the statement.

a. $x + 9 = 9 + x$ **b.** $5(x^3 \cdot 2) = (5x^3)2$ **c.** $(2 + 5x^2)y^2 = 2 \cdot y^2 + 5x^2 \cdot y^2$

REMARK Notice the difference between the *opposite of a number* and a *negative number.* If a is negative, then its opposite, $-a$, is positive. For instance, if $a = -5$, then

$$-a = -(-5) = 5.$$

Properties of Negation and Equality

Let a, b, and c be real numbers, variables, or algebraic expressions.

Property	Example
1. $(-1)a = -a$	$(-1)7 = -7$
2. $-(-a) = a$	$-(-6) = 6$
3. $(-a)b = -(ab) = a(-b)$	$(-5)3 = -(5 \cdot 3) = 5(-3)$
4. $(-a)(-b) = ab$	$(-2)(-x) = 2x$
5. $-(a + b) = (-a) + (-b)$	$-(x + 8) = (-x) + (-8)$ $= -x - 8$
6. If $a = b$, then $a \pm c = b \pm c$.	$\frac{1}{2} + 3 = 0.5 + 3$
7. If $a = b$, then $ac = bc$.	$4^2 \cdot 2 = 16 \cdot 2$
8. If $a \pm c = b \pm c$, then $a = b$.	$1.4 - 1 = \frac{7}{5} - 1$ ⇨ $1.4 = \frac{7}{5}$
9. If $ac = bc$ and $c \neq 0$, then $a = b$.	$3x = 3 \cdot 4$ ⇨ $x = 4$

REMARK The "or" in the Zero-Factor Property includes the possibility that either or both factors may be zero. This is an *inclusive or,* and it is generally the way the word "or" is used in mathematics.

Properties of Zero

Let a and b be real numbers, variables, or algebraic expressions.

1. $a + 0 = a$ and $a - 0 = a$

2. $a \cdot 0 = 0$

3. $\dfrac{0}{a} = 0, \quad a \neq 0$

4. $\dfrac{a}{0}$ is undefined.

5. Zero-Factor Property: If $ab = 0$, then $a = 0$ or $b = 0$.

Properties and Operations of Fractions

Let a, b, c, and d be real numbers, variables, or algebraic expressions such that $b \neq 0$ and $d \neq 0$.

1. **Equivalent Fractions:** $\frac{a}{b} = \frac{c}{d}$ if and only if $ad = bc$.
2. **Rules of Signs:** $-\frac{a}{b} = \frac{-a}{b} = \frac{a}{-b}$ and $\frac{-a}{-b} = \frac{a}{b}$
3. **Generate Equivalent Fractions:** $\frac{a}{b} = \frac{ac}{bc}, \quad c \neq 0$
4. **Add or Subtract with Like Denominators:** $\frac{a}{b} \pm \frac{c}{b} = \frac{a \pm c}{b}$
5. **Add or Subtract with Unlike Denominators:** $\frac{a}{b} \pm \frac{c}{d} = \frac{ad \pm bc}{bd}$
6. **Multiply Fractions:** $\frac{a}{b} \cdot \frac{c}{d} = \frac{ac}{bd}$
7. **Divide Fractions:** $\frac{a}{b} \div \frac{c}{d} = \frac{a}{b} \cdot \frac{d}{c} = \frac{ad}{bc}, \quad c \neq 0$

REMARK In Property 1 of fractions, the phrase "if and only if" implies two statements. One statement is: If $a/b = c/d$, then $ad = bc$. The other statement is: If $ad = bc$, where $b \neq 0$ and $d \neq 0$, then $a/b = c/d$.

EXAMPLE 14 Properties and Operations of Fractions

a. Equivalent fractions: $\frac{x}{5} = \frac{3 \cdot x}{3 \cdot 5} = \frac{3x}{15}$ **b.** Divide fractions: $\frac{7}{x} \div \frac{3}{2} = \frac{7}{x} \cdot \frac{2}{3} = \frac{14}{3x}$

✓ ***Checkpoint*** *Audio-video solution in English & Spanish at LarsonPrecalculus.com.*

a. Multiply fractions: $\frac{3}{5} \cdot \frac{x}{6}$ **b.** Add fractions: $\frac{x}{10} + \frac{2x}{5}$

If a, b, and c are integers such that $ab = c$, then a and b are **factors** or **divisors** of c. A **prime number** is an integer that has exactly two positive factors—itself and 1—such as 2, 3, 5, 7, and 11. The numbers 4, 6, 8, 9, and 10 are **composite** because each can be written as the product of two or more prime numbers. The **Fundamental Theorem of Arithmetic** states that every positive integer greater than 1 is a prime number or can be written as the product of prime numbers in precisely one way (disregarding order). For instance, the *prime factorization* of 24 is $24 = 2 \cdot 2 \cdot 2 \cdot 3$.

REMARK The number 1 is neither prime nor composite.

Summarize (Appendix A.1)

1. Describe how to represent and classify real numbers *(pages A1 and A2)*. For examples of representing and classifying real numbers, see Examples 1 and 2.
2. Describe how to order real numbers and use inequalities *(pages A3 and A4)*. For examples of ordering real numbers and using inequalities, see Examples 3–6.
3. State the absolute value of a real number *(page A5)*. For examples of using absolute value, see Examples 7–10.
4. Explain how to evaluate an algebraic expression *(page A7)*. For examples involving algebraic expressions, see Examples 11 and 12.
5. State the basic rules and properties of algebra *(pages A8–A10)*. For examples involving the basic rules and properties of algebra, see Examples 13 and 14.

A.1 Exercises

See **CalcChat.com** for tutorial help and worked-out solutions to odd-numbered exercises.

Vocabulary: Fill in the blanks.

1. ________ numbers have infinite nonrepeating decimal representations.
2. The point 0 on the real number line is called the ________.
3. The distance between the origin and a point representing a real number on the real number line is the ________ ________ of the real number.
4. A number that can be written as the product of two or more prime numbers is called a ________ number.
5. The ________ of an algebraic expression are those parts separated by addition.
6. The ________ ________ states that if $ab = 0$, then $a = 0$ or $b = 0$.

Skills and Applications

Classifying Real Numbers In Exercises 7–10, determine which numbers in the set are (a) natural numbers, (b) whole numbers, (c) integers, (d) rational numbers, and (e) irrational numbers.

7. $\left\{-9, -\frac{7}{2}, 5, \frac{2}{3}, \sqrt{2}, 0, 1, -4, 2, -11\right\}$

8. $\left\{\sqrt{5}, -7, -\frac{7}{3}, 0, 3.12, \frac{5}{4}, -3, 12, 5\right\}$

9. $\{2.01, 0.666\ldots, -13, 0.010110111\ldots, 1, -6\}$

10. $\left\{25, -17, -\frac{12}{5}, \sqrt{9}, 3.12, \frac{1}{2}\pi, 7, -11.1, 13\right\}$

Plotting Points on the Real Number Line In Exercises 11 and 12, plot the real numbers on the real number line.

11. (a) 3 (b) $\frac{7}{2}$ (c) $-\frac{5}{2}$ (d) -5.2

12. (a) 8.5 (b) $\frac{4}{3}$ (c) -4.75 (d) $-\frac{8}{3}$

Plotting and Ordering Real Numbers In Exercises 13–16, plot the two real numbers on the real number line. Then place the appropriate inequality symbol (< or >) between them.

13. $-4, -8$ **14.** $1, \frac{16}{3}$ **15.** $\frac{5}{6}, \frac{2}{3}$ **16.** $-\frac{8}{7}, -\frac{3}{7}$

Interpreting an Inequality or an Interval In Exercises 17–24, (a) give a verbal description of the subset of real numbers represented by the inequality or the interval, (b) sketch the subset on the real number line, and (c) state whether the interval is bounded or unbounded.

17. $x \le 5$ **18.** $x < 0$

19. $[4, \infty)$ **20.** $(-\infty, 2)$

21. $-2 < x < 2$ **22.** $0 < x \le 6$

23. $[-5, 2)$ **24.** $(-1, 2]$

Using Inequality and Interval Notation In Exercises 25–30, use inequality notation and interval notation to describe the set.

25. y is nonnegative. **26.** y is no more than 25.

27. t is at least 10 and at most 22.

28. k is less than 5 but no less than -3.

29. The dog's weight W is more than 65 pounds.

30. The annual rate of inflation r is expected to be at least 2.5% but no more than 5%.

Evaluating an Absolute Value Expression In Exercises 31–40, evaluate the expression.

31. $|-10|$ **32.** $|0|$

33. $|3 - 8|$ **34.** $|4 - 1|$

35. $|-1| - |-2|$ **36.** $-3 - |-3|$

37. $\dfrac{-5}{|-5|}$ **38.** $-3|-3|$

39. $\dfrac{|x + 2|}{x + 2}, \quad x < -2$ **40.** $\dfrac{|x - 1|}{x - 1}, \quad x > 1$

Comparing Real Numbers In Exercises 41–44, place the appropriate symbol (<, >, or =) between the two real numbers.

41. $|-4| \;\square\; |4|$ **42.** $-5 \;\square\; -|5|$

43. $-|-6| \;\square\; |-6|$ **44.** $-|-2| \;\square\; -|2|$

Finding a Distance In Exercises 45–50, find the distance between *a* and *b*.

45. $a = 126, b = 75$ **46.** $a = -126, b = -75$

47. $a = -\frac{5}{2}, b = 0$ **48.** $a = \frac{1}{4}, b = \frac{11}{4}$

49. $a = \frac{16}{5}, b = \frac{112}{75}$ **50.** $a = 9.34, b = -5.65$

Using Absolute Value Notation In Exercises 51–54, use absolute value notation to describe the situation.

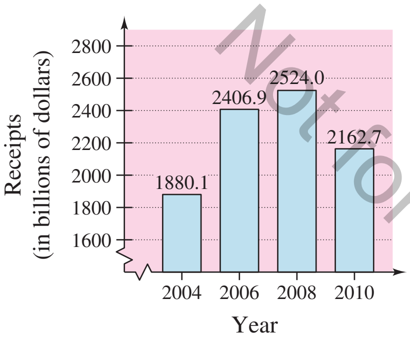
51. The distance between x and 5 is no more than 3.

52. The distance between x and -10 is at least 6.

53. y is at most two units from a.

54. The temperature in Bismarck, North Dakota, was 60°F at noon, then 23°F at midnight. What was the change in temperature over the 12-hour period?

Federal Deficit **In Exercises 55–58, use the bar graph, which shows the receipts of the federal government (in billions of dollars) for selected years from 2004 through 2010. In each exercise you are given the expenditures of the federal government. Find the magnitude of the surplus or deficit for the year.** *(Source: U.S. Office of Management and Budget)*



	Year	Receipts, R	Expenditures, E	$\lvert R - E \rvert$
55.	2004		\$2292.8 billion	
56.	2006		\$2655.1 billion	
57.	2008		\$2982.5 billion	
58.	2010		\$3456.2 billion	

Identifying Terms and Coefficients **In Exercises 59–62, identify the terms. Then identify the coefficients of the variable terms of the expression.**

59. $7x + 4$

60. $6x^3 - 5x$

61. $4x^3 + 0.5x - 5$

62. $3\sqrt{3}x^2 + 1$

Evaluating an Algebraic Expression **In Exercises 63–66, evaluate the expression for each value of x. (If not possible, then state the reason.)**

Expression	Values	
63. $4x - 6$	(a) $x = -1$	(b) $x = 0$
64. $9 - 7x$	(a) $x = -3$	(b) $x = 3$
65. $-x^2 + 5x - 4$	(a) $x = -1$	(b) $x = 1$
66. $(x + 1)/(x - 1)$	(a) $x = 1$	(b) $x = -1$

Identifying Rules of Algebra **In Exercises 67–72, identify the rule(s) of algebra illustrated by the statement.**

67. $\dfrac{1}{h+6}(h + 6) = 1, \quad h \neq -6$

68. $(x + 3) - (x + 3) = 0$

69. $2(x + 3) = 2 \cdot x + 2 \cdot 3$

70. $(z - 2) + 0 = z - 2$

71. $x(3y) = (x \cdot 3)y = (3x)y$

72. $\frac{1}{7}(7 \cdot 12) = \left(\frac{1}{7} \cdot 7\right)12 = 1 \cdot 12 = 12$

Operations with Fractions **In Exercises 73–76, perform the operation(s). (Write fractional answers in simplest form.)**

73. $\frac{5}{8} - \frac{5}{12} + \frac{1}{6}$

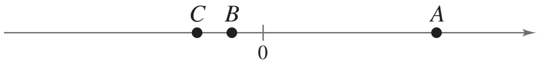
74. $-\left(6 \cdot \frac{4}{8}\right)$

75. $\dfrac{2x}{3} - \dfrac{x}{4}$

76. $\dfrac{5x}{6} \cdot \dfrac{2}{9}$

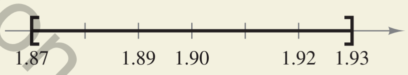
Exploration

77. **Determining the Sign of an Expression** Use the real numbers A, B, and C shown on the number line to determine the sign of (a) $-A$, (b) $B - A$, (c) $-C$, and (d) $A - C$.

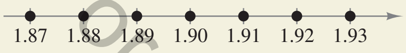


78. **HOW DO YOU SEE IT?** Match each description with its graph. Which types of real numbers shown in Figure A.1 on page A1 may be included in a range of prices? a range of lengths? Explain.

(i)



(ii)



(a) The price of an item is within \$0.03 of \$1.90.

(b) The distance between the prongs of an electric plug may not differ from 1.9 centimeters by more than 0.03 centimeter.

True or False? **In Exercises 79 and 80, determine whether the statement is true or false. Justify your answer.**

79. Every nonnegative number is positive.

80. If $a > 0$ and $b < 0$, then $ab > 0$.

81. **Conjecture**

(a) Use a calculator to complete the table.

n	0.0001	0.01	1	100	10,000
$5/n$					

(b) Use the result from part (a) to make a conjecture about the value of $5/n$ as n (i) approaches 0, and (ii) increases without bound.

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A.2 Exponents and Radicals

Real numbers and algebraic expressions are often written with exponents and radicals. For instance, in Exercise 79 on page A24, you will use an expression involving rational exponents to find the times required for a funnel to empty for different water heights.

- **Use properties of exponents.**
- **Use scientific notation to represent real numbers.**
- **Use properties of radicals.**
- **Simplify and combine radicals.**
- **Rationalize denominators and numerators.**
- **Use properties of rational exponents.**

Integer Exponents

Repeated *multiplication* can be written in **exponential form.**

Repeated Multiplication	**Exponential Form**
$a \cdot a \cdot a \cdot a \cdot a$	a^5
$(-4)(-4)(-4)$	$(-4)^3$
$(2x)(2x)(2x)(2x)$	$(2x)^4$

Exponential Notation

If a is a real number and n is a positive integer, then

$$a^n = \underbrace{a \cdot a \cdot a \cdot \cdot \cdot a}_{n \text{ factors}}$$

where n is the **exponent** and a is the **base.** You read a^n as "a to the nth **power.**"

An exponent can also be negative. In Property 3 below, be sure you see how to use a negative exponent.

Properties of Exponents

Let a and b be real numbers, variables, or algebraic expressions, and let m and n be integers. (All denominators and bases are nonzero.)

Property	**Example**
1. $a^m a^n = a^{m+n}$	$3^2 \cdot 3^4 = 3^{2+4} = 3^6 = 729$
2. $\dfrac{a^m}{a^n} = a^{m-n}$	$\dfrac{x^7}{x^4} = x^{7-4} = x^3$
3. $a^{-n} = \dfrac{1}{a^n} = \left(\dfrac{1}{a}\right)^n$	$y^{-4} = \dfrac{1}{y^4} = \left(\dfrac{1}{y}\right)^4$
4. $a^0 = 1, \quad a \neq 0$	$(x^2 + 1)^0 = 1$
5. $(ab)^m = a^m b^m$	$(5x)^3 = 5^3x^3 = 125x^3$
6. $(a^m)^n = a^{mn}$	$(y^3)^{-4} = y^{3(-4)} = y^{-12} = \dfrac{1}{y^{12}}$
7. $\left(\dfrac{a}{b}\right)^m = \dfrac{a^m}{b^m}$	$\left(\dfrac{2}{x}\right)^3 = \dfrac{2^3}{x^3} = \dfrac{8}{x^3}$
8. $\lvert a^2 \rvert = \lvert a \rvert^2 = a^2$	$\lvert(-2)^2\rvert = \lvert -2 \rvert^2 = (2)^2 = 4$

micropic/iStockphoto.com

TECHNOLOGY You can use a calculator to evaluate exponential expressions. When doing so, it is important to know when to use parentheses because the calculator follows the order of operations. For instance, you would evaluate $(-2)^4$ on a graphing calculator as follows.

((−) 2) ^ 4 ENTER

The display will be 16. If you omit the parentheses, then the display will be -16.

It is important to recognize the difference between expressions such as $(-2)^4$ and -2^4. In $(-2)^4$, the parentheses indicate that the exponent applies to the negative sign as well as to the 2, but in $-2^4 = -(2^4)$, the exponent applies only to the 2. So, $(-2)^4 = 16$ and $-2^4 = -16$.

The properties of exponents listed on the preceding page apply to *all* integers m and n, not just to positive integers, as shown in the examples below.

EXAMPLE 1 Evaluating Exponential Expressions

a. $(-5)^2 = (-5)(-5) = 25$ Negative sign is part of the base.

b. $-5^2 = -(5)(5) = -25$ Negative sign is *not* part of the base.

c. $2 \cdot 2^4 = 2^{1+4} = 2^5 = 32$ Property 1

d. $\dfrac{4^4}{4^6} = 4^{4-6} = 4^{-2} = \dfrac{1}{4^2} = \dfrac{1}{16}$ Properties 2 and 3

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Evaluate each expression.

a. -3^4 **b.** $(-3)^4$

c. $3^2 \cdot 3$ **d.** $\dfrac{3^5}{3^8}$

EXAMPLE 2 Evaluating Algebraic Expressions

Evaluate each algebraic expression when $x = 3$.

a. $5x^{-2}$ **b.** $\frac{1}{3}(-x)^3$

Solution

a. When $x = 3$, the expression $5x^{-2}$ has a value of

$$5x^{-2} = 5(3)^{-2} = \frac{5}{3^2} = \frac{5}{9}.$$

b. When $x = 3$, the expression $\frac{1}{3}(-x)^3$ has a value of

$$\frac{1}{3}(-x)^3 = \frac{1}{3}(-3)^3 = \frac{1}{3}(-27) = -9.$$

✓ **Checkpoint** *Audio-video solution in English & Spanish at LarsonPrecalculus.com.*

Evaluate each algebraic expression when $x = 4$.

a. $-x^{-2}$ **b.** $\frac{1}{4}(-x)^4$

EXAMPLE 3 Using Properties of Exponents

Use the properties of exponents to simplify each expression.

a. $(-3ab^4)(4ab^{-3})$ **b.** $(2xy^2)^3$ **c.** $3a(-4a^2)^0$ **d.** $\left(\frac{5x^3}{y}\right)^2$

Solution

a. $(-3ab^4)(4ab^{-3}) = (-3)(4)(a)(a)(b^4)(b^{-3}) = -12a^2b$

b. $(2xy^2)^3 = 2^3(x)^3(y^2)^3 = 8x^3y^6$

c. $3a(-4a^2)^0 = 3a(1) = 3a, \quad a \neq 0$

d. $\left(\frac{5x^3}{y}\right)^2 = \frac{5^2(x^3)^2}{y^2} = \frac{25x^6}{y^2}$

✓ **Checkpoint** *Audio-video solution in English & Spanish at LarsonPrecalculus.com.*

Use the properties of exponents to simplify each expression.

a. $(2x^{-2}y^3)(-x^4y)$ **b.** $(4a^2b^3)^0$ **c.** $(-5z)^3(z^2)$ **d.** $\left(\frac{3x^4}{x^2y^2}\right)^2$

REMARK Rarely in algebra is there only one way to solve a problem. Do not be concerned when the steps you use to solve a problem are not exactly the same as the steps presented in this text. It is important to use steps that you understand and, of course, steps that are justified by the rules of algebra. For instance, you might prefer the following steps for Example 4(d).

$$\left(\frac{3x^2}{y}\right)^{-2} = \left(\frac{y}{3x^2}\right)^2 = \frac{y^2}{9x^4}$$

Note how the first step of this solution uses Property 3. The fractional form of this property is

$$\left(\frac{a}{b}\right)^{-m} = \left(\frac{b}{a}\right)^m.$$

EXAMPLE 4 Rewriting with Positive Exponents

a. $x^{-1} = \frac{1}{x}$ Property 3

b. $\frac{1}{3x^{-2}} = \frac{1(x^2)}{3}$ The exponent -2 does not apply to 3.

$= \frac{x^2}{3}$ Simplify.

c. $\frac{12a^3b^{-4}}{4a^{-2}b} = \frac{12a^3 \cdot a^2}{4b \cdot b^4}$ Property 3

$= \frac{3a^5}{b^5}$ Property 1

d. $\left(\frac{3x^2}{y}\right)^{-2} = \frac{3^{-2}(x^2)^{-2}}{y^{-2}}$ Properties 5 and 7

$= \frac{3^{-2}x^{-4}}{y^{-2}}$ Property 6

$= \frac{y^2}{3^2x^4}$ Property 3

$= \frac{y^2}{9x^4}$ Simplify.

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Rewrite each expression with positive exponents.

a. $2a^{-2}$ **b.** $\frac{3a^{-3}b^4}{15ab^{-1}}$

c. $\left(\frac{x}{10}\right)^{-1}$ **d.** $(-2x^2)^3(4x^3)^{-1}$

Scientific Notation

Exponents provide an efficient way of writing and computing with very large (or very small) numbers. For instance, there are about 359 billion billion gallons of water on Earth—that is, 359 followed by 18 zeros.

$$359{,}000{,}000{,}000{,}000{,}000{,}000$$

It is convenient to write such numbers in **scientific notation.** This notation has the form $\pm c \times 10^n$, where $1 \leq c < 10$ and n is an integer. So, the number of gallons of water on Earth, written in scientific notation, is

$$3.59 \times 100{,}000{,}000{,}000{,}000{,}000{,}000 = 3.59 \times 10^{20}.$$

The *positive* exponent 20 indicates that the number is *large* (10 or more) and that the decimal point has been moved 20 places. A *negative* exponent indicates that the number is *small* (less than 1). For instance, the mass (in grams) of one electron is approximately

$$9.0 \times 10^{-28} = 0.\underbrace{0000000000000000000000000009}_{\text{28 decimal places}}.$$

EXAMPLE 5 **Scientific Notation**

a. $0.0000782 = 7.82 \times 10^{-5}$

b. $836{,}100{,}000 = 8.361 \times 10^{8}$

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Write 45,850 in scientific notation.

EXAMPLE 6 **Decimal Notation**

a. $-9.36 \times 10^{-6} = -0.00000936$

b. $1.345 \times 10^{2} = 134.5$

✓ ***Checkpoint*** 🔊 *Audio-video solution in English & Spanish at LarsonPrecalculus.com.*

Write -2.718×10^{-3} in decimal notation.

▷ **TECHNOLOGY** Most calculators automatically switch to scientific notation when they are showing large (or small) numbers that exceed the display range.

To enter numbers in scientific notation, your calculator should have an exponential entry key labeled

$\boxed{\text{EE}}$ or $\boxed{\text{EXP}}$.

Consult the user's guide for instructions on keystrokes and how your calculator displays numbers in scientific notation.

EXAMPLE 7 **Using Scientific Notation**

Evaluate $\dfrac{(2{,}400{,}000{,}000)(0.0000045)}{(0.00003)(1500)}$.

Solution Begin by rewriting each number in scientific notation and simplifying.

$$\begin{aligned}\frac{(2{,}400{,}000{,}000)(0.0000045)}{(0.00003)(1500)} &= \frac{(2.4 \times 10^{9})(4.5 \times 10^{-6})}{(3.0 \times 10^{-5})(1.5 \times 10^{3})}\\ &= \frac{(2.4)(4.5)(10^{3})}{(4.5)(10^{-2})}\\ &= (2.4)(10^{5})\\ &= 240{,}000\end{aligned}$$

✓ ***Checkpoint*** 🔊 *Audio-video solution in English & Spanish at LarsonPrecalculus.com.*

Evaluate (24,000,000,000)(0.00000012)(300,000). ■

Radicals and Their Properties

A **square root** of a number is one of its two equal factors. For example, 5 is a square root of 25 because 5 is one of the two equal factors of 25. In a similar way, a **cube root** of a number is one of its three equal factors, as in $125 = 5^3$.

Definition of *n*th Root of a Number

Let a and b be real numbers and let $n \geq 2$ be a positive integer. If

$$a = b^n$$

then b is an ***n*th root of *a*.** When $n = 2$, the root is a **square root.** When $n = 3$, the root is a **cube root.**

Some numbers have more than one nth root. For example, both 5 and -5 are square roots of 25. The *principal square* root of 25, written as $\sqrt{25}$, is the positive root, 5. The **principal *n*th root** of a number is defined as follows.

Principal *n*th Root of a Number

Let a be a real number that has at least one nth root. The **principal *n*th root of *a*** is the nth root that has the same sign as a. It is denoted by a **radical symbol**

$$\sqrt[n]{a}. \qquad \text{Principal } n\text{th root}$$

The positive integer $n \geq 2$ is the **index** of the radical, and the number a is the **radicand.** When $n = 2$, omit the index and write $\sqrt{a}$ rather than $\sqrt[2]{a}$. (The plural of index is *indices.*)

A common misunderstanding is that the square root sign implies both negative and positive roots. This is not correct. The square root sign implies only a positive root. When a negative root is needed, you must use the negative sign with the square root sign.

Incorrect: $\xcancel{\sqrt{4} = \pm 2}$ *Correct:* $-\sqrt{4} = -2$ *and* $\sqrt{4} = 2$

EXAMPLE 8 Evaluating Expressions Involving Radicals

a. $\sqrt{36} = 6$ because $6^2 = 36$.

b. $-\sqrt{36} = -6$ because $-\left(\sqrt{36}\right) = -\left(\sqrt{6^2}\right) = -(6) = -6$.

c. $\sqrt[3]{\dfrac{125}{64}} = \dfrac{5}{4}$ because $\left(\dfrac{5}{4}\right)^3 = \dfrac{5^3}{4^3} = \dfrac{125}{64}$.

d. $\sqrt[5]{-32} = -2$ because $(-2)^5 = -32$.

e. $\sqrt[4]{-81}$ is not a real number because no real number raised to the fourth power produces -81.

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Evaluate each expression (if possible).

a. $-\sqrt{144}$ **b.** $\sqrt{-144}$

c. $\sqrt{\dfrac{25}{64}}$ **d.** $-\sqrt[3]{\dfrac{8}{27}}$

Here are some generalizations about the *n*th roots of real numbers.

Generalizations About *n*th Roots of Real Numbers

Real Number a	Integer $n > 0$	Root(s) of a	Example
$a > 0$	n is even.	$\sqrt[n]{a},\ -\sqrt[n]{a}$	$\sqrt[4]{81} = 3,\ -\sqrt[4]{81} = -3$
$a > 0$ or $a < 0$	n is odd.	$\sqrt[n]{a}$	$\sqrt[3]{-8} = -2$
$a < 0$	n is even.	No real roots	$\sqrt{-4}$ is not a real number.
$a = 0$	n is even or odd.	$\sqrt[n]{0} = 0$	$\sqrt[5]{0} = 0$

Integers such as 1, 4, 9, 16, 25, and 36 are called **perfect squares** because they have integer square roots. Similarly, integers such as 1, 8, 27, 64, and 125 are called **perfect cubes** because they have integer cube roots.

Properties of Radicals

Let a and b be real numbers, variables, or algebraic expressions such that the indicated roots are real numbers, and let m and n be positive integers.

Property	**Example**
1. $\sqrt[n]{a^m} = \left(\sqrt[n]{a}\right)^m$	$\sqrt[3]{8^2} = \left(\sqrt[3]{8}\right)^2 = (2)^2 = 4$
2. $\sqrt[n]{a} \cdot \sqrt[n]{b} = \sqrt[n]{ab}$	$\sqrt{5} \cdot \sqrt{7} = \sqrt{5 \cdot 7} = \sqrt{35}$
3. $\dfrac{\sqrt[n]{a}}{\sqrt[n]{b}} = \sqrt[n]{\dfrac{a}{b}},\quad b \neq 0$	$\dfrac{\sqrt[4]{27}}{\sqrt[4]{9}} = \sqrt[4]{\dfrac{27}{9}} = \sqrt[4]{3}$
4. $\sqrt[m]{\sqrt[n]{a}} = \sqrt[mn]{a}$	$\sqrt[3]{\sqrt{10}} = \sqrt[6]{10}$
5. $\left(\sqrt[n]{a}\right)^n = a$	$\left(\sqrt{3}\right)^2 = 3$
6. For n even, $\sqrt[n]{a^n} = \lvert a \rvert$.	$\sqrt{(-12)^2} = \lvert -12 \rvert = 12$
For n odd, $\sqrt[n]{a^n} = a$.	$\sqrt[3]{(-12)^3} = -12$

A common use of Property 6 is $\sqrt{a^2} = \lvert a \rvert$.

EXAMPLE 9 Using Properties of Radicals

Use the properties of radicals to simplify each expression.

a. $\sqrt{8} \cdot \sqrt{2}$ **b.** $\left(\sqrt[3]{5}\right)^3$

c. $\sqrt[3]{x^3}$ **d.** $\sqrt[6]{y^6}$

Solution

a. $\sqrt{8} \cdot \sqrt{2} = \sqrt{8 \cdot 2} = \sqrt{16} = 4$ **b.** $\left(\sqrt[3]{5}\right)^3 = 5$

c. $\sqrt[3]{x^3} = x$ **d.** $\sqrt[6]{y^6} = \lvert y \rvert$

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Use the properties of radicals to simplify each expression.

a. $\dfrac{\sqrt{125}}{\sqrt{5}}$ **b.** $\sqrt[3]{125^2}$ **c.** $\sqrt[3]{x^2} \cdot \sqrt[3]{x}$ **d.** $\sqrt{\sqrt{x}}$

Simplifying Radicals

An expression involving radicals is in **simplest form** when the following conditions are satisfied.

1. All possible factors have been removed from the radical.
2. All fractions have radical-free denominators (a process called *rationalizing the denominator* accomplishes this).
3. The index of the radical is reduced.

To simplify a radical, factor the radicand into factors whose exponents are multiples of the index. Write the roots of these factors outside the radical. The "leftover" factors make up the new radicand.

REMARK When you simplify a radical, it is important that both expressions are defined for the same values of the variable. For instance, in Example 10(c), $\sqrt{75x^3}$ and $5x\sqrt{3x}$ are both defined only for nonnegative values of x. Similarly, in Example 10(e), $\sqrt[4]{(5x)^4}$ and $5|x|$ are both defined for all real values of x.

EXAMPLE 10 Simplifying Radicals

a. $\sqrt[3]{24} = \sqrt[3]{\overset{\text{Perfect cube}}{8} \cdot \overset{\text{Leftover factor}}{3}} = \sqrt[3]{2^3 \cdot 3} = 2\sqrt[3]{3}$

b. $\sqrt[4]{48} = \sqrt[4]{\overset{\substack{\text{Perfect}\\\text{4th power}}}{16} \cdot \overset{\substack{\text{Leftover}\\\text{factor}}}{3}} = \sqrt[4]{2^4 \cdot 3} = 2\sqrt[4]{3}$

c. $\sqrt{75x^3} = \sqrt{25x^2 \cdot 3x} = \sqrt{(5x)^2 \cdot 3x} = 5x\sqrt{3x}$

d. $\sqrt[3]{24a^4} = \sqrt[3]{8a^3 \cdot 3a} = \sqrt[3]{(2a)^3 \cdot 3a} = 2a\sqrt[3]{3a}$

e. $\sqrt[4]{(5x)^4} = |5x| = 5|x|$

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Simplify each radical expression.

a. $\sqrt{32}$ **b.** $\sqrt[3]{250}$ **c.** $\sqrt{24a^5}$ **d.** $\sqrt[3]{-135x^3}$

Radical expressions can be combined (added or subtracted) when they are **like radicals**—that is, when they have the same index and radicand. For instance, $\sqrt{2}$, $3\sqrt{2}$, and $\frac{1}{2}\sqrt{2}$ are like radicals, but $\sqrt{3}$ and $\sqrt{2}$ are unlike radicals. To determine whether two radicals can be combined, you should first simplify each radical.

EXAMPLE 11 Combining Radicals

a.
$$\begin{aligned} 2\sqrt{48} - 3\sqrt{27} &= 2\sqrt{16 \cdot 3} - 3\sqrt{9 \cdot 3} && \text{Find square factors.}\\ &= 8\sqrt{3} - 9\sqrt{3} && \text{Find square roots and multiply by coefficients.}\\ &= (8 - 9)\sqrt{3} && \text{Combine like radicals.}\\ &= -\sqrt{3} && \text{Simplify.} \end{aligned}$$

b.
$$\begin{aligned} \sqrt[3]{16x} - \sqrt[3]{54x^4} &= \sqrt[3]{8 \cdot 2x} - \sqrt[3]{27 \cdot x^3 \cdot 2x} && \text{Find cube factors.}\\ &= 2\sqrt[3]{2x} - 3x\sqrt[3]{2x} && \text{Find cube roots.}\\ &= (2 - 3x)\sqrt[3]{2x} && \text{Combine like radicals.} \end{aligned}$$

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Simplify each radical expression.

a. $3\sqrt{8} + \sqrt{18}$ **b.** $\sqrt[3]{81x^5} - \sqrt[3]{24x^2}$

Rationalizing Denominators and Numerators

To rationalize a denominator or numerator of the form $a - b\sqrt{m}$ or $a + b\sqrt{m}$, multiply both numerator and denominator by a **conjugate:** $a + b\sqrt{m}$ and $a - b\sqrt{m}$ are conjugates of each other. If $a = 0$, then the rationalizing factor for $\sqrt{m}$ is itself, $\sqrt{m}$. For cube roots, choose a rationalizing factor that generates a perfect cube.

EXAMPLE 12 Rationalizing Single-Term Denominators

Rationalize the denominator of each expression.

a. $\dfrac{5}{2\sqrt{3}}$

b. $\dfrac{2}{\sqrt[3]{5}}$

Solution

a.
$$\begin{aligned}\frac{5}{2\sqrt{3}} &= \frac{5}{2\sqrt{3}} \cdot \frac{\sqrt{3}}{\sqrt{3}} && \sqrt{3}\text{ is rationalizing factor.}\\ &= \frac{5\sqrt{3}}{2(3)} && \text{Multiply.}\\ &= \frac{5\sqrt{3}}{6} && \text{Simplify.}\end{aligned}$$

b.
$$\begin{aligned}\frac{2}{\sqrt[3]{5}} &= \frac{2}{\sqrt[3]{5}} \cdot \frac{\sqrt[3]{5^2}}{\sqrt[3]{5^2}} && \sqrt[3]{5^2}\text{ is rationalizing factor.}\\ &= \frac{2\sqrt[3]{5^2}}{\sqrt[3]{5^3}} && \text{Multiply.}\\ &= \frac{2\sqrt[3]{25}}{5} && \text{Simplify.}\end{aligned}$$

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Rationalize the denominator of each expression.

a. $\dfrac{5}{3\sqrt{2}}$ **b.** $\dfrac{1}{\sqrt[3]{25}}$

EXAMPLE 13 Rationalizing a Denominator with Two Terms

$$\begin{aligned}\frac{2}{3+\sqrt{7}} &= \frac{2}{3+\sqrt{7}} \cdot \frac{3-\sqrt{7}}{3-\sqrt{7}} && \text{Multiply numerator and denominator by conjugate of denominator.}\\ &= \frac{2\left(3-\sqrt{7}\right)}{3(3)+3\left(-\sqrt{7}\right)+\sqrt{7}(3)-\left(\sqrt{7}\right)\left(\sqrt{7}\right)} && \text{Use Distributive Property.}\\ &= \frac{2\left(3-\sqrt{7}\right)}{(3)^2-\left(\sqrt{7}\right)^2} && \text{Simplify.}\\ &= \frac{2\left(3-\sqrt{7}\right)}{2} = 3-\sqrt{7} && \text{Simplify.}\end{aligned}$$

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Rationalize the denominator: $\dfrac{8}{\sqrt{6}-\sqrt{2}}$.

Sometimes it is necessary to rationalize the numerator of an expression. For instance, in Appendix A.4 you will use the technique shown in the next example to rationalize the numerator of an expression from calculus.

EXAMPLE 14 Rationalizing a Numerator

$$\frac{\sqrt{5}-\sqrt{7}}{2} = \frac{\sqrt{5}-\sqrt{7}}{2}\cdot\frac{\sqrt{5}+\sqrt{7}}{\sqrt{5}+\sqrt{7}}$$

Multiply numerator and denominator by conjugate of denominator.

$$= \frac{(\sqrt{5})^2-(\sqrt{7})^2}{2(\sqrt{5}+\sqrt{7})}$$

Simplify.

$$= \frac{5-7}{2(\sqrt{5}+\sqrt{7})}$$

Square terms of numerator.

$$= \frac{-2}{2(\sqrt{5}+\sqrt{7})} = \frac{-1}{\sqrt{5}+\sqrt{7}}$$

Simplify.

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Rationalize the numerator: $\dfrac{2-\sqrt{2}}{3}$.

Rational Exponents

Definition of Rational Exponents

If a is a real number and n is a positive integer such that the principal nth root of a exists, then $a^{1/n}$ is defined as

$a^{1/n} = \sqrt[n]{a}$, where $1/n$ is the **rational exponent** of a.

Moreover, if m is a positive integer that has no common factor with n, then

$$a^{m/n} = (a^{1/n})^m = \left(\sqrt[n]{a}\right)^m \quad\text{and}\quad a^{m/n} = (a^m)^{1/n} = \sqrt[n]{a^m}.$$

The numerator of a rational exponent denotes the *power* to which the base is raised, and the denominator denotes the *index* or the *root* to be taken.

Power

Index

$$b^{m/n} = \left(\sqrt[n]{b}\right)^m = \sqrt[n]{b^m}$$

REMARK You must remember that the expression $b^{m/n}$ is not defined unless $\sqrt[n]{b}$ is a real number. This restriction produces some unusual results. For instance, the number $(-8)^{1/3}$ is defined because $\sqrt[3]{-8} = -2$, but the number $(-8)^{2/6}$ is undefined because $\sqrt[6]{-8}$ is not a real number.

When you are working with rational exponents, the properties of integer exponents still apply. For instance, $2^{1/2}2^{1/3} = 2^{(1/2)+(1/3)} = 2^{5/6}$.

EXAMPLE 15 Changing From Radical to Exponential Form

a. $\sqrt{3} = 3^{1/2}$ **b.** $\sqrt{(3xy)^5} = \sqrt[2]{(3xy)^5} = (3xy)^{5/2}$

c. $2x\sqrt[4]{x^3} = (2x)(x^{3/4}) = 2x^{1+(3/4)} = 2x^{7/4}$

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Write each expression in exponential form.

a. $\sqrt[3]{27}$ **b.** $\sqrt{x^3y^5z}$ **c.** $3x\sqrt[3]{x^2}$

TECHNOLOGY There are four methods of evaluating radicals on most graphing calculators. For square roots, you can use the *square root key* $\sqrt{\ }$. For cube roots, you can use the *cube root key* $\sqrt[3]{\ }$. For other roots, you can first convert the radical to exponential form and then use the *exponential key* ^, or you can use the xth root key $\sqrt[x]{\ }$ (or menu choice). Consult the user's guide for your calculator for specific keystrokes.

EXAMPLE 16 Changing From Exponential to Radical Form

a. $(x^2 + y^2)^{3/2} = \left(\sqrt{x^2 + y^2}\right)^3 = \sqrt{(x^2 + y^2)^3}$

b. $2y^{3/4}z^{1/4} = 2(y^3z)^{1/4} = 2\sqrt[4]{y^3z}$

c. $a^{-3/2} = \dfrac{1}{a^{3/2}} = \dfrac{1}{\sqrt{a^3}}$

d. $x^{0.2} = x^{1/5} = \sqrt[5]{x}$

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Write each expression in radical form.

a. $(x^2 - 7)^{-1/2}$ **b.** $-3b^{1/3}c^{2/3}$

c. $a^{0.75}$ **d.** $(x^2)^{2/5}$ ■

Rational exponents are useful for evaluating roots of numbers on a calculator, for reducing the index of a radical, and for simplifying expressions in calculus.

EXAMPLE 17 Simplifying with Rational Exponents

a. $(-32)^{-4/5} = \left(\sqrt[5]{-32}\right)^{-4} = (-2)^{-4} = \dfrac{1}{(-2)^4} = \dfrac{1}{16}$

b. $(-5x^{5/3})(3x^{-3/4}) = -15x^{(5/3)-(3/4)} = -15x^{11/12}, \quad x \neq 0$

c. $\sqrt[9]{a^3} = a^{3/9} = a^{1/3} = \sqrt[3]{a}$ Reduce index.

d. $\sqrt[3]{\sqrt{125}} = \sqrt[6]{125} = \sqrt[6]{(5)^3} = 5^{3/6} = 5^{1/2} = \sqrt{5}$

e. $(2x - 1)^{4/3}(2x - 1)^{-1/3} = (2x - 1)^{(4/3)-(1/3)} = 2x - 1, \quad x \neq \frac{1}{2}$

REMARK The expression in Example 17(e) is not defined when $x = \frac{1}{2}$ because

$$\left(2 \cdot \frac{1}{2} - 1\right)^{-1/3} = (0)^{-1/3}$$

is not a real number.

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Simplify each expression.

a. $(-125)^{-2/3}$ **b.** $(4x^2y^{3/2})(-3x^{-1/3}y^{-3/5})$

c. $\sqrt[3]{\sqrt[4]{27}}$ **d.** $(3x + 2)^{5/2}(3x + 2)^{-1/2}$ ■

Summarize (Appendix A.2)

1. Make a list of the properties of exponents *(page A13)*. For examples that use these properties, see Examples 1–4.
2. State the definition of scientific notation *(page A16)*. For examples of scientific notation, see Examples 5–7.
3. Make a list of the properties of radicals *(page A18)*. For an example that uses these properties, see Example 9.
4. Explain how to simplify a radical expression *(page A19)*. For examples of simplifying radical expressions, see Examples 10 and 11.
5. Explain how to rationalize a denominator or a numerator *(page A20)*. For examples of rationalizing denominators and numerators, see Examples 12–14.
6. State the definition of a rational exponent *(page A21)*. For an example of simplifying expressions with rational exponents, see Example 17.

A.2 Exercises

See **CalcChat.com** for tutorial help and worked-out solutions to odd-numbered exercises.

Vocabulary: Fill in the blanks.

1. In the exponential form a^n, n is the ________ and a is the ________.
2. A convenient way of writing very large or very small numbers is called ________ ________.
3. One of the two equal factors of a number is called a __________ __________ of the number.
4. In the radical form $\sqrt[n]{a}$, the positive integer n is the ________ of the radical and the number a is the ________.
5. Radical expressions can be combined (added or subtracted) when they are ________ ________.
6. The expressions $a + b\sqrt{m}$ and $a - b\sqrt{m}$ are ________ of each other.
7. The process used to create a radical-free denominator is known as ________ the denominator.
8. In the expression $b^{m/n}$, m denotes the ________ to which the base is raised and n denotes the ________ or root to be taken.

Skills and Applications

Evaluating Exponential Expressions **In Exercises 9–14, evaluate each expression.**

9. (a) $3 \cdot 3^3$ (b) $\dfrac{3^2}{3^4}$

10. (a) $(3^3)^0$ (b) -3^2

11. (a) $(2^3 \cdot 3^2)^2$ (b) $\left(-\frac{3}{5}\right)^3\left(\frac{5}{3}\right)^2$

12. (a) $\dfrac{3}{3^{-4}}$ (b) $48(-4)^{-3}$

13. (a) $\dfrac{4 \cdot 3^{-2}}{2^{-2} \cdot 3^{-1}}$ (b) $(-2)^0$

14. (a) $3^{-1} + 2^{-2}$ (b) $(3^{-2})^2$

Evaluating an Algebraic Expression **In Exercises 15–20, evaluate the expression for the given value of x.**

15. $-3x^3, \quad x = 2$

16. $7x^{-2}, \quad x = 4$

17. $6x^0, \quad x = 10$

18. $2x^3, \quad x = -3$

19. $-3x^4, \quad x = -2$

20. $12(-x)^3, \quad x = -\frac{1}{3}$

Using Properties of Exponents **In Exercises 21–26, simplify each expression.**

21. (a) $(-5z)^3$ (b) $5x^4(x^2)$

22. (a) $(3x)^2$ (b) $(4x^3)^0, \quad x \neq 0$

23. (a) $6y^2(2y^0)^2$ (b) $(-z)^3(3z^4)$

24. (a) $\dfrac{7x^2}{x^3}$ (b) $\dfrac{12(x + y)^3}{9(x + y)}$

25. (a) $\left(\dfrac{4}{y}\right)^3\left(\dfrac{3}{y}\right)^4$ (b) $\left(\dfrac{b^{-2}}{a^{-2}}\right)\left(\dfrac{b}{a}\right)^2$

26. (a) $[(x^2y^{-2})^{-1}]^{-1}$ (b) $(5x^2z^6)^3(5x^2z^6)^{-3}$

Rewriting with Positive Exponents **In Exercises 27–30, rewrite each expression with positive exponents and simplify.**

27. (a) $(x + 5)^0, \quad x \neq -5$ (b) $(2x^2)^{-2}$

28. (a) $(4y^{-2})(8y^4)$ (b) $(z + 2)^{-3}(z + 2)^{-1}$

29. (a) $\left(\dfrac{x^{-3}y^4}{5}\right)^{-3}$ (b) $\left(\dfrac{a^{-2}}{b^{-2}}\right)\left(\dfrac{b}{a}\right)^3$

30. (a) $3^n \cdot 3^{2n}$ (b) $\dfrac{x^2 \cdot x^n}{x^3 \cdot x^n}$

Scientific Notation **In Exercises 31–34, write the number in scientific notation.**

31. 10,250.4

32. -0.000125

33. One micron (millionth of a meter): 0.00003937 inch

34. Land area of Earth: 57,300,000 square miles

Decimal Notation **In Exercises 35–38, write the number in decimal notation.**

35. 3.14×10^{-4}

36. -1.801×10^5

37. Light year: 9.46×10^{12} kilometers

38. Width of a human hair: 9.0×10^{-5} meter

Using Scientific Notation **In Exercises 39 and 40, evaluate each expression without using a calculator.**

39. (a) $(2.0 \times 10^9)(3.4 \times 10^{-4})$
(b) $(1.2 \times 10^7)(5.0 \times 10^{-3})$

40. (a) $\dfrac{6.0 \times 10^8}{3.0 \times 10^{-3}}$ (b) $\dfrac{2.5 \times 10^{-3}}{5.0 \times 10^2}$

Evaluating Expressions Involving Radicals **In Exercises 41 and 42, evaluate each expression without using a calculator.**

41. (a) $\sqrt{9}$ (b) $\sqrt[3]{\frac{27}{8}}$

42. (a) $\sqrt[3]{27}$ (b) $\left(\sqrt{36}\right)^3$

Using Properties of Radicals **In Exercises 43 and 44, use the properties of radicals to simplify each expression.**

43. (a) $\left(\sqrt[5]{2}\right)^5$ (b) $\sqrt[5]{32x^5}$

44. (a) $\sqrt{12} \cdot \sqrt{3}$ (b) $\sqrt[4]{(3x^2)^4}$

Simplifying Radical Expressions **In Exercises 45–54, simplify each radical expression.**

45. (a) $\sqrt{20}$ (b) $\sqrt[3]{128}$

46. (a) $\sqrt[3]{\frac{16}{27}}$ (b) $\sqrt{\frac{75}{4}}$

47. (a) $\sqrt{72x^3}$ (b) $\sqrt{54xy^4}$

48. (a) $\sqrt{\dfrac{18^2}{z^3}}$ (b) $\sqrt{\dfrac{32a^4}{b^2}}$

49. (a) $\sqrt[3]{16x^5}$ (b) $\sqrt{75x^2y^{-4}}$

50. (a) $\sqrt[4]{3x^4y^2}$ (b) $\sqrt[5]{160x^8z^4}$

51. (a) $10\sqrt{32} - 6\sqrt{18}$ (b) $\sqrt[3]{16} + 3\sqrt[3]{54}$

52. (a) $5\sqrt{x} - 3\sqrt{x}$ (b) $-2\sqrt{9y} + 10\sqrt{y}$

53. (a) $-3\sqrt{48x^2} + 7\sqrt{75x^2}$

(b) $7\sqrt{80x} - 2\sqrt{125x}$

54. (a) $-\sqrt{x^3 - 7} + 5\sqrt{x^3 - 7}$

(b) $11\sqrt{245x^3} - 9\sqrt{45x^3}$

Rationalizing a Denominator **In Exercises 55–58, rationalize the denominator of the expression. Then simplify your answer.**

55. $\dfrac{1}{\sqrt{3}}$

56. $\dfrac{8}{\sqrt[3]{2}}$

57. $\dfrac{5}{\sqrt{14} - 2}$

58. $\dfrac{3}{\sqrt{5} + \sqrt{6}}$

Rationalizing a Numerator **In Exercises 59–62, rationalize the numerator of the expression. Then simplify your answer.**

59. $\dfrac{\sqrt{8}}{2}$

60. $\dfrac{\sqrt{2}}{3}$

61. $\dfrac{\sqrt{5} + \sqrt{3}}{3}$

62. $\dfrac{\sqrt{7} - 3}{4}$

Writing Exponential and Radical Forms **In Exercises 63–68, fill in the missing form of the expression.**

	Radical Form	**Rational Exponent Form**
63.	$\sqrt[3]{64}$	
64.		$(2ab)^{3/4}$
65.		$3x^{-2/3}$
66.	$x^2\sqrt{x}$	
67.	$x\sqrt{3xy}$	
68.		$a^{0.4}$

Simplifying Expressions **In Exercises 69–78, simplify each expression.**

69. (a) $32^{-3/5}$ (b) $\left(\frac{16}{81}\right)^{-3/4}$

70. (a) $100^{-3/2}$ (b) $\left(\frac{9}{4}\right)^{-1/2}$

71. (a) $(2x^2)^{3/2} \cdot 2^{-1/2} \cdot x^{-4}$ (b) $(x^4y^2)^{1/3}(xy)^{-1/3}$

72. (a) $x^{-3} \cdot x^{1/2} \cdot x^{-3/2} \cdot x$ (b) $5^{-1/2} \cdot 5x^{5/2}(5x)^{-3/2}$

73. (a) $\sqrt[4]{3^2}$ (b) $\sqrt[6]{(x + 1)^4}$

74. (a) $\sqrt[6]{x^3}$ (b) $\sqrt[4]{(3x^2)^4}$

75. (a) $\sqrt{\sqrt{32}}$ (b) $\sqrt{\sqrt[4]{2x}}$

76. (a) $\sqrt{\sqrt{243(x + 1)}}$ (b) $\sqrt{\sqrt[3]{10a^7b}}$

77. (a) $(x - 1)^{1/3}(x - 1)^{2/3}$

(b) $(x - 1)^{1/3}(x - 1)^{-4/3}$

78. (a) $(4x + 3)^{5/2}(4x + 3)^{-5/3}$

(b) $(4x + 3)^{-5/2}(4x + 3)^{1/2}$

79. Mathematical Modeling

A funnel is filled with water to a height of h centimeters. The formula

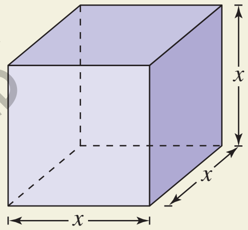
$$t = 0.03[12^{5/2} - (12 - h)^{5/2}], \quad 0 \le h \le 12$$

represents the amount of time t (in seconds) that it will take for the funnel to empty.

(a) Use the *table* feature of a graphing utility to find the times required for the funnel to empty for water heights of $h = 0, h = 1, h = 2, \ldots, h = 12$ centimeters.

(b) What value does t appear to be approaching as the height of the water becomes closer and closer to 12 centimeters?

80. **HOW DO YOU SEE IT?** Package A is a cube with a volume of 500 cubic inches. Package B is a cube with a volume of 250 cubic inches. Is the length x of a side of package A greater than, less than, or equal to twice the length of a side of package B? Explain.



Exploration

True or False? **In Exercises 81–84, determine whether the statement is true or false. Justify your answer.**

81. $\dfrac{x^{k+1}}{x} = x^k$

82. $(a^n)^k = a^{n^k}$

83. $(a + b)^2 = a^2 + b^2$

84. $\dfrac{a}{\sqrt{b}} = \dfrac{a^2}{(\sqrt{b})^2} = \dfrac{a^2}{b}$

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A.3 Polynomials and Factoring

Polynomial factoring can help you solve real-life problems. For instance, in Exercise 96 on page A34, you will use factoring to develop an alternative model for the rate of change of an autocatalytic chemical reaction.

- **Write polynomials in standard form.**
- **Add, subtract, and multiply polynomials.**
- **Use special products to multiply polynomials.**
- **Remove common factors from polynomials.**
- **Factor special polynomial forms.**
- **Factor trinomials as the product of two binomials.**
- **Factor polynomials by grouping.**

Polynomials

One of the most common types of algebraic expressions is the **polynomial.** Some examples are $2x + 5$, $3x^4 - 7x^2 + 2x + 4$, and $5x^2y^2 - xy + 3$. The first two are *polynomials in x* and the third is a *polynomial in x and y*. The terms of a polynomial in x have the form ax^k, where a is the **coefficient** and k is the **degree** of the term. For instance, the polynomial $2x^3 - 5x^2 + 1 = 2x^3 + (-5)x^2 + (0)x + 1$ has coefficients $2, -5, 0,$ and 1.

Definition of a Polynomial in *x*

Let $a_0, a_1, a_2, \ldots, a_n$ be real numbers and let n be a nonnegative integer. A polynomial in x is an expression of the form

$$a_nx^n + a_{n-1}x^{n-1} + \cdot \cdot \cdot + a_1x + a_0$$

where $a_n \neq 0$. The polynomial is of **degree** n, a_n is the **leading coefficient,** and a_0 is the **constant term.**

REMARK Expressions are not polynomials when a variable is underneath a radical or when a polynomial expression (with degree greater than 0) is in the denominator of a term. For example, the expressions $x^3 - \sqrt{3x} = x^3 - (3x)^{1/2}$ and $x^2 + (5/x) = x^2 + 5x^{-1}$ are not polynomials.

Polynomials with one, two, and three terms are called **monomials, binomials,** and **trinomials,** respectively. A polynomial written with descending powers of x is in **standard form.**

EXAMPLE 1 Writing Polynomials in Standard Form

				Leading
	Polynomial	**Standard Form**	**Degree**	**Coefficient**
a.	$4x^2 - 5x^7 - 2 + 3x$	$-5x^7 + 4x^2 + 3x - 2$	7	-5
b.	$4 - 9x^2$	$-9x^2 + 4$	2	-9
c.	8	$8 \quad (8 = 8x^0)$	0	8

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Write the polynomial $6 - 7x^3 + 2x$ in standard form. Then identify the degree and leading coefficient of the polynomial.

A polynomial that has all zero coefficients is called the **zero polynomial,** denoted by 0. No degree is assigned to the zero polynomial. For polynomials in more than one variable, the degree of a *term* is the sum of the exponents of the variables in the term. The degree of the *polynomial* is the highest degree of its terms. For instance, the degree of the polynomial $-2x^3y^6 + 4xy - x^7y^4$ is 11 because the sum of the exponents in the last term is the greatest. The leading coefficient of the polynomial is the coefficient of the highest-degree term.

mikeledray/Shutterstock.com

Operations with Polynomials

You can add and subtract polynomials in much the same way you add and subtract real numbers. Add or subtract the *like terms* (terms having the same variables to the same powers) by adding or subtracting their coefficients. For instance, $-3xy^2$ and $5xy^2$ are like terms and their sum is

$$-3xy^2 + 5xy^2 = (-3 + 5)xy^2 = 2xy^2.$$

EXAMPLE 2 Sums and Differences of Polynomials

a. $(5x^3 - 7x^2 - 3) + (x^3 + 2x^2 - x + 8)$

$= (5x^3 + x^3) + (-7x^2 + 2x^2) - x + (-3 + 8)$ Group like terms.

$= 6x^3 - 5x^2 - x + 5$ Combine like terms.

b. $(7x^4 - x^2 - 4x + 2) - (3x^4 - 4x^2 + 3x)$

$= 7x^4 - x^2 - 4x + 2 - 3x^4 + 4x^2 - 3x$ Distributive Property

$= (7x^4 - 3x^4) + (-x^2 + 4x^2) + (-4x - 3x) + 2$ Group like terms.

$= 4x^4 + 3x^2 - 7x + 2$ Combine like terms.

REMARK When a negative sign precedes an expression inside parentheses, remember to distribute the negative sign to each term inside the parentheses, as shown.

$$-(3x^4 - 4x^2 + 3x)$$
$$= -3x^4 + 4x^2 - 3x$$

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Find the difference $(2x^3 - x + 3) - (x^2 - 2x - 3)$ and write the resulting polynomial in standard form.

To find the *product* of two polynomials, use the left and right Distributive Properties. For example, if you treat $5x + 7$ as a single quantity, then you can multiply $3x - 2$ by $5x + 7$ as follows.

$$(3x - 2)(5x + 7) = 3x(5x + 7) - 2(5x + 7)$$
$$= (3x)(5x) + (3x)(7) - (2)(5x) - (2)(7)$$
$$= \underbrace{15x^2}_{\text{Product of First terms}} + \underbrace{21x}_{\text{Product of Outer terms}} - \underbrace{10x}_{\text{Product of Inner terms}} - \underbrace{14}_{\text{Product of Last terms}}$$
$$= 15x^2 + 11x - 14$$

Note in this **FOIL Method** (which can only be used to multiply two binomials) that the outer (O) and inner (I) terms are like terms and can be combined.

EXAMPLE 3 Finding a Product by the FOIL Method

Use the FOIL Method to find the product of $2x - 4$ and $x + 5$.

Solution

$$(2x - 4)(x + 5) = \overset{\text{F}}{2x^2} + \overset{\text{O}}{10x} - \overset{\text{I}}{4x} - \overset{\text{L}}{20}$$
$$= 2x^2 + 6x - 20$$

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Use the FOIL Method to find the product of $3x - 1$ and $x - 5$.

Special Products

Some binomial products have special forms that occur frequently in algebra. You do not need to memorize these formulas because you can use the Distributive Property to multiply. However, becoming familiar with these formulas will enable you to manipulate the algebra more quickly.

Special Products

Let u and v be real numbers, variables, or algebraic expressions.

Special Product	**Example**
Sum and Difference of Same Terms	
$(u + v)(u - v) = u^2 - v^2$	$(x + 4)(x - 4) = x^2 - 4^2$ $= x^2 - 16$
Square of a Binomial	
$(u + v)^2 = u^2 + 2uv + v^2$	$(x + 3)^2 = x^2 + 2(x)(3) + 3^2$ $= x^2 + 6x + 9$
$(u - v)^2 = u^2 - 2uv + v^2$	$(3x - 2)^2 = (3x)^2 - 2(3x)(2) + 2^2$ $= 9x^2 - 12x + 4$
Cube of a Binomial	
$(u + v)^3 = u^3 + 3u^2v + 3uv^2 + v^3$	$(x + 2)^3 = x^3 + 3x^2(2) + 3x(2^2) + 2^3$ $= x^3 + 6x^2 + 12x + 8$
$(u - v)^3 = u^3 - 3u^2v + 3uv^2 - v^3$	$(x - 1)^3 = x^3 - 3x^2(1) + 3x(1^2) - 1^3$ $= x^3 - 3x^2 + 3x - 1$

EXAMPLE 4 Sum and Difference of Same Terms

Find each product.

a. $(5x + 9)(5x - 9)$ **b.** $(x + y - 2)(x + y + 2)$

Solution

a. The product of a sum and a difference of the *same* two terms has no middle term and takes the form $(u + v)(u - v) = u^2 - v^2$.

$$(5x + 9)(5x - 9) = (5x)^2 - 9^2 = 25x^2 - 81$$

b. By grouping $x + y$ in parentheses, you can write the product of the trinomials as a special product.

$$\begin{aligned}(x + y - 2)(x + y + 2) &= [(x + y) \overset{\substack{\text{Difference}\\ \downarrow}}{-} 2][(x + y) \overset{\substack{\text{Sum}\\ \downarrow}}{+} 2]\\ &= (x + y)^2 - 2^2 \qquad \text{Sum and difference of same terms}\\ &= x^2 + 2xy + y^2 - 4\end{aligned}$$

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Find the product: $(x - 2 + 3y)(x - 2 - 3y)$.

Polynomials with Common Factors

The process of writing a polynomial as a product is called **factoring.** It is an important tool for solving equations and for simplifying rational expressions.

Unless noted otherwise, when you are asked to factor a polynomial, assume that you are looking for factors that have integer coefficients. If a polynomial does not factor using integer coefficients, then it is **prime** or **irreducible over the integers.** For instance, the polynomial

$$x^2 - 3$$

is irreducible over the integers. Over the *real numbers,* this polynomial factors as

$$x^2 - 3 = (x + \sqrt{3})(x - \sqrt{3}).$$

A polynomial is **completely factored** when each of its factors is prime. For instance,

$$x^3 - x^2 + 4x - 4 = (x - 1)(x^2 + 4) \qquad \text{Completely factored}$$

is completely factored, but

$$x^3 - x^2 - 4x + 4 = (x - 1)(x^2 - 4) \qquad \text{Not completely factored}$$

is not completely factored. Its complete factorization is

$$x^3 - x^2 - 4x + 4 = (x - 1)(x + 2)(x - 2).$$

The simplest type of factoring involves a polynomial that can be written as the product of a monomial and another polynomial. The technique used here is the Distributive Property, $a(b + c) = ab + ac$, in the *reverse* direction.

$$ab + ac = a(b + c) \qquad a \text{ is a common factor.}$$

Removing (factoring out) any common factors is the first step in completely factoring a polynomial.

EXAMPLE 5 Removing Common Factors

Factor each expression.

a. $6x^3 - 4x$

b. $-4x^2 + 12x - 16$

c. $(x - 2)(2x) + (x - 2)(3)$

Solution

a. $6x^3 - 4x = 2x(3x^2) - 2x(2)$ $\qquad 2x$ is a common factor.

$= 2x(3x^2 - 2)$

b. $-4x^2 + 12x - 16 = -4(x^2) + (-4)(-3x) + (-4)4$ $\qquad -4$ is a common factor.

$= -4(x^2 - 3x + 4)$

c. $(x - 2)(2x) + (x - 2)(3) = (x - 2)(2x + 3)$ $\qquad (x - 2)$ is a common factor.

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Factor each expression.

a. $5x^3 - 15x^2$

b. $-3 + 6x - 12x^3$

c. $(x + 1)(x^2) - (x + 1)(2)$

Factoring Special Polynomial Forms

Some polynomials have special forms that arise from the special product forms on page A27. You should learn to recognize these forms.

Factoring Special Polynomial Forms

Factored Form	**Example**
Difference of Two Squares	
$u^2 - v^2 = (u + v)(u - v)$	$9x^2 - 4 = (3x)^2 - 2^2 = (3x + 2)(3x - 2)$
Perfect Square Trinomial	
$u^2 + 2uv + v^2 = (u + v)^2$	$x^2 + 6x + 9 = x^2 + 2(x)(3) + 3^2 = (x + 3)^2$
$u^2 - 2uv + v^2 = (u - v)^2$	$x^2 - 6x + 9 = x^2 - 2(x)(3) + 3^2 = (x - 3)^2$
Sum or Difference of Two Cubes	
$u^3 + v^3 = (u + v)(u^2 - uv + v^2)$	$x^3 + 8 = x^3 + 2^3 = (x + 2)(x^2 - 2x + 4)$
$u^3 - v^3 = (u - v)(u^2 + uv + v^2)$	$27x^3 - 1 = (3x)^3 - 1^3 = (3x - 1)(9x^2 + 3x + 1)$

The factored form of a difference of two squares is always a set of *conjugate pairs.*

$$u^2 - v^2 = (u + v)(u - v)$$

Conjugate pairs

Difference

Opposite signs

To recognize perfect square terms, look for coefficients that are squares of integers and variables raised to *even powers.*

EXAMPLE 6 Removing a Common Factor First

$$3 - 12x^2 = 3(1 - 4x^2) \qquad \text{3 is a common factor.}$$

$$= 3[1^2 - (2x)^2]$$

$$= 3(1 + 2x)(1 - 2x) \qquad \text{Difference of two squares}$$

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Factor $100 - 4y^2$.

REMARK In Example 6, note that the first step in factoring a polynomial is to check for any common factors. Once you have removed any common factors, it is often possible to recognize patterns that were not immediately obvious.

EXAMPLE 7 Factoring the Difference of Two Squares

a. $(x + 2)^2 - y^2 = [(x + 2) + y][(x + 2) - y]$

$$= (x + 2 + y)(x + 2 - y)$$

b. $16x^4 - 81 = (4x^2)^2 - 9^2$

$$= (4x^2 + 9)(4x^2 - 9) \qquad \text{Difference of two squares}$$

$$= (4x^2 + 9)[(2x)^2 - 3^2]$$

$$= (4x^2 + 9)(2x + 3)(2x - 3) \qquad \text{Difference of two squares}$$

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Factor $(x - 1)^2 - 9y^4$.

A **perfect square trinomial** is the square of a binomial, and it has the form

$$u^2 \underbrace{+ 2uv + v^2 = (u +}_{\text{Like signs}} v)^2 \quad \text{or} \quad u^2 \underbrace{- 2uv + v^2 = (u -}_{\text{Like signs}} v)^2.$$

Note that the first and last terms are squares and the middle term is twice the product of u and v.

EXAMPLE 8 Factoring Perfect Square Trinomials

Factor each trinomial.

a. $x^2 - 10x + 25$ **b.** $16x^2 + 24x + 9$

Solution

a. $x^2 - 10x + 25 = x^2 - 2(x)(5) + 5^2 = (x - 5)^2$

b. $16x^2 + 24x + 9 = (4x)^2 + 2(4x)(3) + 3^2 = (4x + 3)^2$

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Factor $9x^2 - 30x + 25$.

The next two formulas show the sums and differences of cubes. Pay special attention to the signs of the terms.

$$u^3 \overbrace{+ v^3 = (u}^{\text{Like signs}} \underbrace{+ v)(u^2 -}_{\text{Unlike signs}} uv + v^2) \qquad u^3 \overbrace{- v^3 = (u}^{\text{Like signs}} \underbrace{- v)(u^2 +}_{\text{Unlike signs}} uv + v^2)$$

EXAMPLE 9 Factoring the Difference of Cubes

$$\begin{aligned} x^3 - 27 &= x^3 - 3^3 && \text{Rewrite 27 as } 3^3. \\ &= (x - 3)(x^2 + 3x + 9) && \text{Factor.} \end{aligned}$$

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Factor $64x^3 - 1$.

EXAMPLE 10 Factoring the Sum of Cubes

a.
$$\begin{aligned} y^3 + 8 &= y^3 + 2^3 && \text{Rewrite 8 as } 2^3. \\ &= (y + 2)(y^2 - 2y + 4) && \text{Factor.} \end{aligned}$$

b.
$$\begin{aligned} 3x^3 + 192 &= 3(x^3 + 64) && \text{3 is a common factor.} \\ &= 3(x^3 + 4^3) && \text{Rewrite 64 as } 4^3. \\ &= 3(x + 4)(x^2 - 4x + 16) && \text{Factor.} \end{aligned}$$

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Factor each expression.

a. $x^3 + 216$ **b.** $5y^3 + 135$

Trinomials with Binomial Factors

To factor a trinomial of the form $ax^2 + bx + c$, use the following pattern.

Factors of a

$$ax^2 + bx + c = (\square x + \square)(\square x + \square)$$

Factors of c

The goal is to find a combination of factors of a and c such that the outer and inner products add up to the middle term bx. For instance, for the trinomial $6x^2 + 17x + 5$, you can write all possible factorizations and determine which one has outer and inner products that add up to $17x$.

$$(6x + 5)(x + 1),\ (6x + 1)(x + 5),\ (2x + 1)(3x + 5),\ (2x + 5)(3x + 1)$$

You can see that $(2x + 5)(3x + 1)$ is the correct factorization because the outer (O) and inner (I) products add up to $17x$.

$$(2x + 5)(3x + 1) = \overset{\text{F}}{\overset{\downarrow}{6x^2}} + \overset{\text{O}}{\overset{\downarrow}{2x}} + \overset{\text{I}}{\overset{\downarrow}{15x}} + \overset{\text{L}}{\overset{\downarrow}{5}} = 6x^2 + \overset{\text{O + I}}{\overset{\downarrow}{17x}} + 5$$

EXAMPLE 11 Factoring a Trinomial: Leading Coefficient Is 1

Factor $x^2 - 7x + 12$.

Solution For this trinomial, you have $a = 1$, $b = -7$, and $c = 12$. Because b is negative and c is positive, both factors of 12 must be negative. So, the possible factorizations of $x^2 - 7x + 12$ are

$$(x - 2)(x - 6),\quad (x - 1)(x - 12),\quad \text{and}\quad (x - 3)(x - 4).$$

Testing the middle term, you will find the correct factorization to be

$$x^2 - 7x + 12 = (x - 3)(x - 4).$$

REMARK Factoring a trinomial can involve trial and error. However, once you have produced the factored form, it is relatively easy to check your answer. For instance, verify the factorization in Example 11 by multiplying $(x - 3)$ by $(x - 4)$ to see that you obtain the original trinomial $x^2 - 7x + 12$.

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Factor $x^2 + x - 6$.

EXAMPLE 12 Factoring a Trinomial: Leading Coefficient Is Not 1

Factor $2x^2 + x - 15$.

Solution For this trinomial, you have $a = 2$ and $c = -15$, which means that the factors of -15 must have unlike signs. The eight possible factorizations are as follows.

$$(2x - 1)(x + 15)\quad (2x + 1)(x - 15)\quad (2x - 3)(x + 5)\quad (2x + 3)(x - 5)$$

$$(2x - 5)(x + 3)\quad (2x + 5)(x - 3)\quad (2x - 15)(x + 1)\quad (2x + 15)(x - 1)$$

Testing the middle term, you will find the correct factorization to be

$$2x^2 + x - 15 = (2x - 5)(x + 3).\qquad \text{O + I} = 6x - 5x = x$$

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Factor each trinomial.

a. $2x^2 - 5x + 3$ **b.** $12x^2 + 7x + 1$

Factoring by Grouping

Sometimes, polynomials with more than three terms can be **factored by grouping.**

EXAMPLE 13 Factoring by Grouping

$$\begin{aligned} x^3 - 2x^2 - 3x + 6 &= (x^3 - 2x^2) - (3x - 6) && \text{Group terms.} \\ &= x^2(x - 2) - 3(x - 2) && \text{Factor each group.} \\ &= (x - 2)(x^2 - 3) && \text{Distributive Property} \end{aligned}$$

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Factor $x^3 + x^2 - 5x - 5$.

REMARK Sometimes, more than one grouping will work. For instance, another way to factor the polynomial in Example 13 is as follows.

$$\begin{aligned} &x^3 - 2x^2 - 3x + 6 \\ &= (x^3 - 3x) - (2x^2 - 6) \\ &= x(x^2 - 3) - 2(x^2 - 3) \\ &= (x^2 - 3)(x - 2) \end{aligned}$$

As you can see, you obtain the same result as in Example 13.

Factoring by grouping can save you some of the trial and error involved in factoring a trinomial. To factor a trinomial of the form $ax^2 + bx + c$ by grouping, rewrite the middle term using the sum of two factors of the product ac that add up to b. Example 14 illustrates this technique.

EXAMPLE 14 Factoring a Trinomial by Grouping

In the trinomial $2x^2 + 5x - 3$, $a = 2$ and $c = -3$, so the product ac is -6. Now, -6 factors as $(6)(-1)$ and $6 - 1 = 5 = b$. So, rewrite the middle term as $5x = 6x - x$.

$$\begin{aligned} 2x^2 + 5x - 3 &= 2x^2 + 6x - x - 3 && \text{Rewrite middle term.} \\ &= (2x^2 + 6x) - (x + 3) && \text{Group terms.} \\ &= 2x(x + 3) - (x + 3) && \text{Factor groups.} \\ &= (x + 3)(2x - 1) && \text{Distributive Property} \end{aligned}$$

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Use factoring by grouping to factor $2x^2 + 5x - 12$.

Summarize (Appendix A.3)

1. State the definition of a polynomial in x and explain what is meant by the standard form of a polynomial *(page A25)*. For an example of writing polynomials in standard form, see Example 1.
2. Describe how to add and subtract polynomials *(page A26)*. For an example of adding and subtracting polynomials, see Example 2.
3. Describe the FOIL Method *(page A26)*. For an example of finding a product using the FOIL Method, see Example 3.
4. Explain how to find binomial products that have special forms *(page A27)*. For an example of binomial products that have special forms, see Example 4.
5. Describe what it means to completely factor a polynomial *(page A28)*. For an example of removing common factors, see Example 5.
6. Make a list of the special polynomial forms of factoring *(page A29)*. For examples of factoring these special forms, see Examples 6–10.
7. Describe how to factor a trinomial of the form $ax^2 + bx + c$ *(page A31)*. For examples of factoring trinomials of this form, see Examples 11 and 12.
8. Explain how to factor a polynomial by grouping *(page A32)*. For examples of factoring by grouping, see Examples 13 and 14.

A.3 Exercises

See CalcChat.com for tutorial help and worked-out solutions to odd-numbered exercises.

Vocabulary: Fill in the blanks.

1. For the polynomial $a_nx^n + a_{n-1}x^{n-1} + \cdot \cdot \cdot + a_1x + a_0$, $a_n \neq 0$, the degree is ________, the leading coefficient is ________, and the constant term is ________.

2. A polynomial with one term is called a ________, while a polynomial with two terms is called a ________ and a polynomial with three terms is called a ________.

3. To add or subtract polynomials, add or subtract the ________ ________ by adding their coefficients.

4. The letters in “FOIL” stand for the following. F ________ O ________ I ________ L ________

5. The process of writing a polynomial as a product is called ________.

6. A polynomial is ________ ________ when each of its factors is prime.

7. A ________ ________ ________ is the square of a binomial, and it has the form $u^2 + 2uv + v^2$ or $u^2 - 2uv + v^2$.

8. When a polynomial has more than three terms, a method of factoring called ________ ________ ________ may be used.

Skills and Applications

Polynomials **In Exercises 9–18, (a) write the polynomial in standard form, (b) identify the degree and leading coefficient of the polynomial, and (c) state whether the polynomial is a monomial, a binomial, or a trinomial.**

9. $14x - \frac{1}{2}x^5$
10. $7x$
11. $3 - x^6$
12. $-y + 25y^2 + 1$
13. 3
14. $-8 + t^2$
15. $1 + 6x^4 - 4x^5$
16. $3 + 2x$
17. $4x^3y$
18. $-x^5y + 2x^2y^2 + xy^4$

Operations with Polynomials **In Exercises 19–26, perform the operation and write the result in standard form.**

19. $(6x + 5) - (8x + 15)$
20. $(2x^2 + 1) - (x^2 - 2x + 1)$
21. $(15x^2 - 6) - (-8.3x^3 - 14.7x^2 - 17)$
22. $(15.6w^4 - 14w - 17.4) - (16.9w^4 - 9.2w + 13)$
23. $3x(x^2 - 2x + 1)$
24. $y^2(4y^2 + 2y - 3)$
25. $-5z(3z - 1)$
26. $(-3x)(5x + 2)$

Multiplying Polynomials **In Exercises 27–40, multiply or find the special product.**

27. $(x + 3)(x + 4)$
28. $(x - 5)(x + 10)$
29. $(x^2 - x + 1)(x^2 + x + 1)$
30. $(2x^2 - x + 4)(x^2 + 3x + 2)$
31. $(x + 10)(x - 10)$
32. $(4a + 5b)(4a - 5b)$
33. $(2x + 3)^2$
34. $(8x + 3)^2$
35. $(x + 1)^3$
36. $(3x + 2y)^3$
37. $[(m - 3) + n][(m - 3) - n]$
38. $[(x - 3y) + z][(x - 3y) - z]$
39. $[(x - 3) + y]^2$
40. $[(x + 1) - y]^2$

Factoring Out a Common Factor **In Exercises 41–44, factor out the common factor.**

41. $2x^3 - 6x$
42. $3z^3 - 6z^2 + 9z$
43. $3x(x - 5) + 8(x - 5)$
44. $(x + 3)^2 - 4(x + 3)$

Greatest Common Factor **In Exercises 45–48, find the greatest common factor such that the remaining factors have only integer coefficients.**

45. $\frac{1}{2}x^3 + 2x^2 - 5x$
46. $\frac{1}{3}y^4 - 5y^2 + 2y$
47. $\frac{2}{3}x(x - 3) - 4(x - 3)$
48. $\frac{4}{5}y(y + 1) - 2(y + 1)$

Factoring the Difference of Two Squares **In Exercises 49–52, completely factor the difference of two squares.**

49. $x^2 - 81$
50. $x^2 - 64$
51. $(x - 1)^2 - 4$
52. $25 - (z + 5)^2$

Factoring a Perfect Square Trinomial **In Exercises 53–58, factor the perfect square trinomial.**

53. $x^2 - 4x + 4$
54. $4t^2 + 4t + 1$
55. $9u^2 + 24uv + 16v^2$
56. $36y^2 - 108y + 81$
57. $z^2 + z + \frac{1}{4}$
58. $9y^2 - \frac{3}{2}y + \frac{1}{16}$

Factoring the Sum or Difference of Cubes **In Exercises 59–62, factor the sum or difference of cubes.**

59. $x^3 - 8$
60. $27 - x^3$
61. $27x^3 + 8$
62. $u^3 + 27v^3$

Factoring a Trinomial **In Exercises 63–70, factor the trinomial.**

63. $x^2 + x - 2$

64. $s^2 - 5s + 6$

65. $20 - y - y^2$

66. $24 + 5z - z^2$

67. $3x^2 - 5x + 2$

68. $2x^2 - x - 1$

69. $5x^2 + 26x + 5$

70. $-9z^2 + 3z + 2$

Factoring by Grouping **In Exercises 71–76, factor by grouping.**

71. $x^3 - x^2 + 2x - 2$

72. $x^3 + 5x^2 - 5x - 25$

73. $2x^3 - x^2 - 6x + 3$

74. $6 + 2x - 3x^3 - x^4$

75. $x^5 + 2x^3 + x^2 + 2$

76. $8x^5 - 6x^2 + 12x^3 - 9$

Factoring a Trinomial by Grouping **In Exercises 77–80, factor the trinomial by grouping.**

77. $2x^2 + 9x + 9$

78. $6x^2 + x - 2$

79. $6x^2 - x - 15$

80. $12x^2 - 13x + 1$

Factoring Completely **In Exercises 81–94, completely factor the expression.**

81. $6x^2 - 54$

82. $12x^2 - 48$

83. $x^3 - x^2$

84. $x^3 - 16x$

85. $x^2 - 2x + 1$

86. $16 + 6x - x^2$

87. $2x^2 + 4x - 2x^3$

88. $13x + 6 + 5x^2$

89. $5 - x + 5x^2 - x^3$

90. $3u - 2u^2 + 6 - u^3$

91. $5(3 - 4x)^2 - 8(3 - 4x)(5x - 1)$

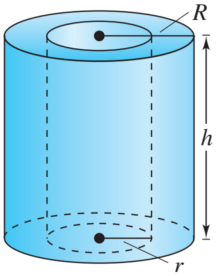
92. $2(x + 1)(x - 3)^2 - 3(x + 1)^2(x - 3)$

93. $x^4(4)(2x + 1)^3(2x) + (2x + 1)^4(4x^3)$

94. $x^3(3)(x^2 + 1)^2(2x) + (x^2 + 1)^3(3x^2)$

95. Geometry The cylindrical shell shown in the figure has a volume of

$$V = \pi R^2 h - \pi r^2 h.$$



(a) Factor the expression for the volume.

(b) From the result of part (a), show that the volume is $2\pi(\text{average radius})(\text{thickness of the shell})h$.

96. Chemistry The rate of change of an autocatalytic chemical reaction is $kQx - kx^2$, where Q is the amount of the original substance, x is the amount of substance formed, and k is a constant of proportionality. Factor the expression.

Exploration

True or False? **In Exercises 97–100, determine whether the statement is true or false. Justify your answer.**

97. The product of two binomials is always a second-degree polynomial.

98. The sum of two binomials is always a binomial.

99. The difference of two perfect squares can be factored as the product of conjugate pairs.

100. The sum of two perfect squares can be factored as the binomial sum squared.

101. Degree of a Product Find the degree of the product of two polynomials of degrees m and n.

102. Degree of a Sum Find the degree of the sum of two polynomials of degrees m and n, where $m < n$.

103. Think About It When the polynomial

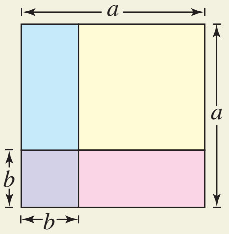
$$-x^3 + 3x^2 + 2x - 1$$

is subtracted from an unknown polynomial, the difference is $5x^2 + 8$. If it is possible, then find the unknown polynomial.

104. Logical Reasoning Verify that $(x + y)^2$ is not equal to $x^2 + y^2$ by letting $x = 3$ and $y = 4$ and evaluating both expressions. Are there any values of x and y for which $(x + y)^2 = x^2 + y^2$? Explain.

105. Think About It Give an example of a polynomial that is prime with respect to the integers.

106. HOW DO YOU SEE IT? The figure shows a large square with an area of a^2 that contains a smaller square with an area of b^2.



(a) Describe the regions that represent $a^2 - b^2$. How can you rearrange these regions to show that $a^2 - b^2 = (a - b)(a + b)$?

(b) How can you use the figure to show that $(a - b)^2 = a^2 - 2ab + b^2$?

(c) Draw another figure to show that $(a + b)^2 = a^2 + 2ab + b^2$. Explain how the figure shows this.

Factoring with Variables in the Exponents **In Exercises 107 and 108, factor the expression as completely as possible.**

107. $x^{2n} - y^{2n}$

108. $x^{3n} + y^{3n}$

mikeledray/Shutterstock.com

A.4 Rational Expressions

Rational expressions can help you solve real-life problems. For instance, in Exercise 73 on page A43, you will use a rational expression to model the temperature of food in a refrigerator.

- **Find domains of algebraic expressions.**
- **Simplify rational expressions.**
- **Add, subtract, multiply, and divide rational expressions.**
- **Simplify complex fractions and rewrite difference quotients.**

Domain of an Algebraic Expression

The set of real numbers for which an algebraic expression is defined is the **domain** of the expression. Two algebraic expressions are **equivalent** when they have the same domain and yield the same values for all numbers in their domain. For instance,

$$(x + 1) + (x + 2) \quad \text{and} \quad 2x + 3$$

are equivalent because

$$\begin{aligned}(x + 1) + (x + 2) &= x + 1 + x + 2\\ &= x + x + 1 + 2\\ &= 2x + 3.\end{aligned}$$

EXAMPLE 1 Finding the Domain of an Algebraic Expression

a. The domain of the polynomial

$$2x^3 + 3x + 4$$

is the set of all real numbers. In fact, the domain of any polynomial is the set of all real numbers, unless the domain is specifically restricted.

b. The domain of the radical expression

$$\sqrt{x - 2}$$

is the set of real numbers greater than or equal to 2, because the square root of a negative number is not a real number.

c. The domain of the expression

$$\frac{x + 2}{x - 3}$$

is the set of all real numbers except $x = 3$, which would result in division by zero, which is undefined.

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Find the domain of each expression.

a. $4x^3 + 3, \quad x \geq 0$ **b.** $\sqrt{x + 7}$ **c.** $\dfrac{1 - x}{x}$

The quotient of two algebraic expressions is a *fractional expression.* Moreover, the quotient of two *polynomials* such as

$$\frac{1}{x}, \quad \frac{2x - 1}{x + 1}, \quad \text{or} \quad \frac{x^2 - 1}{x^2 + 1}$$

is a **rational expression.**

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Simplifying Rational Expressions

Recall that a fraction is in simplest form when its numerator and denominator have no factors in common aside from ±1. To write a fraction in simplest form, divide out common factors.

$$\frac{a \cdot \cancel{c}}{b \cdot \cancel{c}} = \frac{a}{b}, \quad c \neq 0$$

The key to success in simplifying rational expressions lies in your ability to *factor* polynomials. When simplifying rational expressions, factor each polynomial completely to determine whether the numerator and denominator have factors in common.

EXAMPLE 2 Simplifying a Rational Expression

$$\frac{x^2 + 4x - 12}{3x - 6} = \frac{(x + 6)\cancel{(x - 2)}}{3\cancel{(x - 2)}} \qquad \text{Factor completely.}$$

$$= \frac{x + 6}{3}, \quad x \neq 2 \qquad \text{Divide out common factor.}$$

Note that the original expression is undefined when $x = 2$ (because division by zero is undefined). To make the simplified expression *equivalent* to the original expression, you must list the domain restriction $x \neq 2$ with the simplified expression.

REMARK In Example 2, do not make the mistake of trying to simplify further by dividing out terms.

$$\xcancel{\frac{x + 6}{3} = \frac{x + \cancel{6}}{\cancel{3}} = x + 2}$$

Remember that to simplify fractions, divide out common *factors,* not terms.

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Write $\dfrac{4x + 12}{x^2 - 3x - 18}$ in simplest form. ■

Sometimes it may be necessary to change the sign of a factor by factoring out (-1) to simplify a rational expression, as shown in Example 3.

EXAMPLE 3 Simplifying a Rational Expression

$$\frac{12 + x - x^2}{2x^2 - 9x + 4} = \frac{(4 - x)(3 + x)}{(2x - 1)(x - 4)} \qquad \text{Factor completely.}$$

$$= \frac{-\cancel{(x - 4)}(3 + x)}{(2x - 1)\cancel{(x - 4)}} \qquad (4 - x) = -(x - 4)$$

$$= -\frac{3 + x}{2x - 1}, \quad x \neq 4 \qquad \text{Divide out common factor.}$$

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Write $\dfrac{3x^2 - x - 2}{5 - 4x - x^2}$ in simplest form. ■

In this text, when writing a rational expression, the domain is usually not listed with the expression. It is *implied* that the real numbers that make the denominator zero are excluded from the expression. Also, when performing operations with rational expressions, this text follows the convention of listing *by the simplified expression* all values of x that must be specifically excluded from the domain in order to make the domains of the simplified and original expressions agree. Example 3, for instance, lists the restriction $x \neq 4$ with the simplified expression to make the two domains agree. Note that the value $x = \frac{1}{2}$ is excluded from *both* domains, so it is not necessary to list this value.

Operations with Rational Expressions

To multiply or divide rational expressions, use the properties of fractions discussed in Appendix A.1. Recall that to divide fractions, you invert the divisor and multiply.

EXAMPLE 4 Multiplying Rational Expressions

$$\frac{2x^2+x-6}{x^2+4x-5}\cdot\frac{x^3-3x^2+2x}{4x^2-6x}=\frac{\cancel{(2x-3)}(x+2)}{(x+5)\cancel{(x-1)}}\cdot\frac{\cancel{x}(x-2)\cancel{(x-1)}}{2\cancel{x}\cancel{(2x-3)}}$$

$$=\frac{(x+2)(x-2)}{2(x+5)},\quad x\neq 0,\ x\neq 1,\ x\neq \tfrac{3}{2}$$

REMARK Note that Example 4 lists the restrictions $x \neq 0$, $x \neq 1$, and $x \neq \frac{3}{2}$ with the simplified expression in order to make the two domains agree. Also note that the value $x = -5$ is excluded from both domains, so it is not necessary to list this value.

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Multiply and simplify: $\dfrac{15x^2+5x}{x^3-3x^2-18x}\cdot\dfrac{x^2-2x-15}{3x^2-8x-3}.$

EXAMPLE 5 Dividing Rational Expressions

$$\frac{x^3-8}{x^2-4}\div\frac{x^2+2x+4}{x^3+8}=\frac{x^3-8}{x^2-4}\cdot\frac{x^3+8}{x^2+2x+4}$$ Invert and multiply.

$$=\frac{\cancel{(x-2)}\cancel{(x^2+2x+4)}}{\cancel{(x+2)}\cancel{(x-2)}}\cdot\frac{\cancel{(x+2)}(x^2-2x+4)}{\cancel{(x^2+2x+4)}}$$

$$=x^2-2x+4,\quad x\neq\pm 2$$ Divide out common factors.

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Divide and simplify: $\dfrac{x^3-1}{x^2-1}\div\dfrac{x^2+x+1}{x^2+2x+1}.$

To add or subtract rational expressions, use the LCD (least common denominator) method or the *basic definition*

$$\frac{a}{b}\pm\frac{c}{d}=\frac{ad\pm bc}{bd},\quad b\neq 0,\ d\neq 0.$$ Basic definition

This definition provides an efficient way of adding or subtracting *two* fractions that have no common factors in their denominators.

EXAMPLE 6 Subtracting Rational Expressions

$$\frac{x}{x-3}-\frac{2}{3x+4}=\frac{x(3x+4)-2(x-3)}{(x-3)(3x+4)}$$ Basic definition

$$=\frac{3x^2+4x-2x+6}{(x-3)(3x+4)}$$ Distributive Property

$$=\frac{3x^2+2x+6}{(x-3)(3x+4)}$$ Combine like terms.

REMARK When subtracting rational expressions, remember to distribute the negative sign to all the terms in the quantity that is being subtracted.

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Subtract: $\dfrac{x}{2x-1}-\dfrac{1}{x+2}.$

For three or more fractions, or for fractions with a repeated factor in the denominators, the LCD method works well. Recall that the least common denominator of several fractions consists of the product of all prime factors in the denominators, with each factor given the highest power of its occurrence in any denominator. Here is a numerical example.

$$\begin{aligned}
\frac{1}{6}+\frac{3}{4}-\frac{2}{3} &= \frac{1\cdot 2}{6\cdot 2}+\frac{3\cdot 3}{4\cdot 3}-\frac{2\cdot 4}{3\cdot 4} && \text{The LCD is 12.}\\
&= \frac{2}{12}+\frac{9}{12}-\frac{8}{12}\\
&= \frac{3}{12}\\
&= \frac{1}{4}
\end{aligned}$$

Sometimes, the numerator of the answer has a factor in common with the denominator. In such cases, simplify the answer. For instance, in the example above, $\frac{3}{12}$ simplifies to $\frac{1}{4}$.

EXAMPLE 7 Combining Rational Expressions: The LCD Method

Perform the operations and simplify.

$$\frac{3}{x-1}-\frac{2}{x}+\frac{x+3}{x^2-1}$$

Solution Using the factored denominators

$$(x-1),\quad x,\quad \text{and}\quad (x+1)(x-1)$$

you can see that the LCD is $x(x+1)(x-1)$.

$$\begin{aligned}
&\frac{3}{x-1}-\frac{2}{x}+\frac{x+3}{(x+1)(x-1)}\\
&= \frac{3(x)(x+1)}{x(x+1)(x-1)}-\frac{2(x+1)(x-1)}{x(x+1)(x-1)}+\frac{(x+3)(x)}{x(x+1)(x-1)}\\
&= \frac{3(x)(x+1)-2(x+1)(x-1)+(x+3)(x)}{x(x+1)(x-1)}\\
&= \frac{3x^2+3x-2x^2+2+x^2+3x}{x(x+1)(x-1)} && \text{Distributive Property}\\
&= \frac{3x^2-2x^2+x^2+3x+3x+2}{x(x+1)(x-1)} && \text{Group like terms.}\\
&= \frac{2x^2+6x+2}{x(x+1)(x-1)} && \text{Combine like terms.}\\
&= \frac{2(x^2+3x+1)}{x(x+1)(x-1)} && \text{Factor.}
\end{aligned}$$

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Perform the operations and simplify.

$$\frac{4}{x}-\frac{x+5}{x^2-4}+\frac{4}{x+2}$$

Complex Fractions and the Difference Quotient

Fractional expressions with separate fractions in the numerator, denominator, or both are called **complex fractions.** Here are two examples.

$$\frac{\left(\frac{1}{x}\right)}{x^2+1} \quad \text{and} \quad \frac{\left(\frac{1}{x}\right)}{\left(\frac{1}{x^2+1}\right)}$$

To simplify a complex fraction, combine the fractions in the numerator into a single fraction and then combine the fractions in the denominator into a single fraction. Then invert the denominator and multiply.

EXAMPLE 8 Simplifying a Complex Fraction

$$\frac{\left(\frac{2}{x}-3\right)}{\left(1-\frac{1}{x-1}\right)} = \frac{\left[\frac{2-3(x)}{x}\right]}{\left[\frac{1(x-1)-1}{x-1}\right]} \qquad \text{Combine fractions.}$$

$$= \frac{\left(\frac{2-3x}{x}\right)}{\left(\frac{x-2}{x-1}\right)} \qquad \text{Simplify.}$$

$$= \frac{2-3x}{x} \cdot \frac{x-1}{x-2} \qquad \text{Invert and multiply.}$$

$$= \frac{(2-3x)(x-1)}{x(x-2)}, \quad x \neq 1$$

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Simplify the complex fraction $\frac{\left(\frac{1}{x+2}+1\right)}{\left(\frac{x}{3}-1\right)}$. ■

Another way to simplify a complex fraction is to multiply its numerator and denominator by the LCD of all fractions in its numerator and denominator. This method is applied to the fraction in Example 8 as follows.

$$\frac{\left(\frac{2}{x}-3\right)}{\left(1-\frac{1}{x-1}\right)} = \frac{\left(\frac{2}{x}-3\right)}{\left(1-\frac{1}{x-1}\right)} \cdot \frac{x(x-1)}{x(x-1)} \qquad \text{LCD is } x(x-1).$$

$$= \frac{\left(\frac{2-3x}{\cancel{x}}\right)\cdot \cancel{x}(x-1)}{\left(\frac{x-2}{\cancel{x-1}}\right)\cdot x(\cancel{x-1})}$$

$$= \frac{(2-3x)(x-1)}{x(x-2)}, \quad x \neq 1$$

The next three examples illustrate some methods for simplifying rational expressions involving negative exponents and radicals. These types of expressions occur frequently in calculus.

To simplify an expression with negative exponents, one method is to begin by factoring out the common factor with the *lesser* exponent. Remember that when factoring, you *subtract* exponents. For instance, in $3x^{-5/2} + 2x^{-3/2}$, the lesser exponent is $-\frac{5}{2}$ and the common factor is $x^{-5/2}$.

$$\begin{aligned} 3x^{-5/2} + 2x^{-3/2} &= x^{-5/2}[3(1) + 2x^{-3/2-(-5/2)}] \\ &= x^{-5/2}(3 + 2x^{1}) \\ &= \frac{3 + 2x}{x^{5/2}} \end{aligned}$$

EXAMPLE 9 Simplifying an Expression

Simplify the following expression containing negative exponents.

$$x(1 - 2x)^{-3/2} + (1 - 2x)^{-1/2}$$

Solution Begin by factoring out the common factor with the *lesser exponent.*

$$\begin{aligned} x(1 - 2x)^{-3/2} + (1 - 2x)^{-1/2} &= (1 - 2x)^{-3/2}[x + (1 - 2x)^{(-1/2)-(-3/2)}] \\ &= (1 - 2x)^{-3/2}[x + (1 - 2x)^{1}] \\ &= \frac{1 - x}{(1 - 2x)^{3/2}} \end{aligned}$$

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Simplify $(x - 1)^{-1/3} - x(x - 1)^{-4/3}$.

The next example shows a second method for simplifying an expression with negative exponents.

EXAMPLE 10 Simplifying an Expression

$$\begin{aligned} &\frac{(4 - x^2)^{1/2} + x^2(4 - x^2)^{-1/2}}{4 - x^2} \\ &\qquad = \frac{(4 - x^2)^{1/2} + x^2(4 - x^2)^{-1/2}}{4 - x^2} \cdot \frac{(4 - x^2)^{1/2}}{(4 - x^2)^{1/2}} \\ &\qquad = \frac{(4 - x^2)^{1} + x^2(4 - x^2)^{0}}{(4 - x^2)^{3/2}} \\ &\qquad = \frac{4 - x^2 + x^2}{(4 - x^2)^{3/2}} \\ &\qquad = \frac{4}{(4 - x^2)^{3/2}} \end{aligned}$$

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Simplify

$$\frac{x^2(x^2 - 2)^{-1/2} + (x^2 - 2)^{1/2}}{x^2 - 2}.$$

EXAMPLE 11 Rewriting a Difference Quotient

The following expression from calculus is an example of a *difference quotient.*

$$\frac{\sqrt{x+h}-\sqrt{x}}{h}$$

Rewrite this expression by rationalizing its numerator.

Solution

$$\begin{aligned}\frac{\sqrt{x+h}-\sqrt{x}}{h} &= \frac{\sqrt{x+h}-\sqrt{x}}{h}\cdot\frac{\sqrt{x+h}+\sqrt{x}}{\sqrt{x+h}+\sqrt{x}}\\ &= \frac{\left(\sqrt{x+h}\right)^2-\left(\sqrt{x}\right)^2}{h\left(\sqrt{x+h}+\sqrt{x}\right)}\\ &= \frac{x+h-x}{h\left(\sqrt{x+h}+\sqrt{x}\right)}\\ &= \frac{h}{h\left(\sqrt{x+h}+\sqrt{x}\right)}\\ &= \frac{1}{\sqrt{x+h}+\sqrt{x}}, \quad h \neq 0\end{aligned}$$

ALGEBRA HELP You can review the techniques for rationalizing a numerator in Appendix A.2.

Notice that the original expression is undefined when $h = 0$. So, you must exclude $h = 0$ from the domain of the simplified expression so that the expressions are equivalent.

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Rewrite the difference quotient

$$\frac{\sqrt{9+h}-3}{h}$$

by rationalizing its numerator.

Difference quotients, such as that in Example 11, occur frequently in calculus. Often, they need to be rewritten in an equivalent form that can be evaluated when $h = 0$.

Summarize (Appendix A.4)

1. State the definition of the domain of an algebraic expression *(page A35)*. For an example of finding the domains of algebraic expressions, see Example 1.
2. State the definition of a rational expression and describe how to simplify a rational expression *(pages A35 and A36)*. For examples of simplifying rational expressions, see Examples 2 and 3.
3. Describe how to multiply, divide, add, and subtract rational expressions *(page A37)*. For examples of operations with rational expressions, see Examples 4–7.
4. State the definition of a complex fraction *(page A39)*. For an example of simplifying a complex fraction, see Example 8.
5. Describe how to rewrite a difference quotient and why you would want to do so *(page A41)*. For an example of rewriting a difference quotient, see Example 11.

A.4 Exercises

See **CalcChat.com** for tutorial help and worked-out solutions to odd-numbered exercises.

Vocabulary: Fill in the blanks.

1. The set of real numbers for which an algebraic expression is defined is the ________ of the expression.

2. The quotient of two algebraic expressions is a fractional expression, and the quotient of two polynomials is a ________ ________.

3. Fractional expressions with separate fractions in the numerator, denominator, or both are called ________ fractions.

4. Two algebraic expressions that have the same domain and yield the same values for all numbers in their domains are called ________.

Skills and Applications

Finding the Domain of an Algebraic Expression

In Exercises 5–16, find the domain of the expression.

5. $3x^2 - 4x + 7$

6. $6x^2 - 9, \quad x > 0$

7. $\dfrac{1}{3 - x}$

8. $\dfrac{x + 6}{3x + 2}$

9. $\dfrac{x^2 - 1}{x^2 - 2x + 1}$

10. $\dfrac{x^2 - 5x + 6}{x^2 - 4}$

11. $\dfrac{x^2 - 2x - 3}{x^2 - 6x + 9}$

12. $\dfrac{x^2 - x - 12}{x^2 - 8x + 16}$

13. $\sqrt{4 - x}$

14. $\sqrt{2x - 5}$

15. $\dfrac{1}{\sqrt{x - 3}}$

16. $\dfrac{1}{\sqrt{x + 2}}$

Simplifying a Rational Expression

In Exercises 17–30, write the rational expression in simplest form.

17. $\dfrac{15x^2}{10x}$

18. $\dfrac{18y^2}{60y^5}$

19. $\dfrac{3xy}{xy + x}$

20. $\dfrac{4y - 8y^2}{10y - 5}$

21. $\dfrac{x - 5}{10 - 2x}$

22. $\dfrac{12 - 4x}{x - 3}$

23. $\dfrac{y^2 - 16}{y + 4}$

24. $\dfrac{x^2 - 25}{5 - x}$

25. $\dfrac{x^3 + 5x^2 + 6x}{x^2 - 4}$

26. $\dfrac{x^2 + 8x - 20}{x^2 + 11x + 10}$

27. $\dfrac{2 - x + 2x^2 - x^3}{x^2 - 4}$

28. $\dfrac{x^2 - 9}{x^3 + x^2 - 9x - 9}$

29. $\dfrac{z^3 - 8}{z^2 + 2z + 4}$

30. $\dfrac{y^3 - 2y^2 - 3y}{y^3 + 1}$

31. **Error Analysis** Describe the error.

$$\xcancel{\frac{5x^3}{2x^3 + 4} = \frac{5\cancel{x^3}}{2\cancel{x^3} + 4} = \frac{5}{2 + 4} = \frac{5}{6}}$$

32. **Evaluating a Rational Expression** Complete the table. What can you conclude?

x	0	1	2	3	4	5	6
$\dfrac{x - 3}{x^2 - x - 6}$							
$\dfrac{1}{x + 2}$							

Multiplying or Dividing Rational Expressions

In Exercises 33–38, perform the multiplication or division and simplify.

33. $\dfrac{5}{x - 1} \cdot \dfrac{x - 1}{25(x - 2)}$

34. $\dfrac{r}{r - 1} \div \dfrac{r^2}{r^2 - 1}$

35. $\dfrac{4y - 16}{5y + 15} \div \dfrac{4 - y}{2y + 6}$

36. $\dfrac{t^2 - t - 6}{t^2 + 6t + 9} \cdot \dfrac{t + 3}{t^2 - 4}$

37. $\dfrac{x^2 + xy - 2y^2}{x^3 + x^2y} \cdot \dfrac{x}{x^2 + 3xy + 2y^2}$

38. $\dfrac{x^2 - 14x + 49}{x^2 - 49} \div \dfrac{3x - 21}{x + 7}$

Adding or Subtracting Rational Expressions

In Exercises 39–46, perform the addition or subtraction and simplify.

39. $6 - \dfrac{5}{x + 3}$

40. $\dfrac{2x - 1}{x + 3} + \dfrac{1 - x}{x + 3}$

41. $\dfrac{3}{x - 2} + \dfrac{5}{2 - x}$

42. $\dfrac{2x}{x - 5} - \dfrac{5}{5 - x}$

43. $\dfrac{4}{2x + 1} - \dfrac{x}{x + 2}$

44. $\dfrac{1}{x^2 - x - 2} - \dfrac{x}{x^2 - 5x + 6}$

45. $-\dfrac{1}{x} + \dfrac{2}{x^2 + 1} + \dfrac{1}{x^3 + x}$

46. $\dfrac{2}{x + 1} + \dfrac{2}{x - 1} + \dfrac{1}{x^2 - 1}$

Error Analysis **In Exercises 47 and 48, describe the error.**

47. $$\frac{x+4}{x+2} - \frac{3x-8}{x+2} = \frac{x+4-3x-8}{x+2} = \frac{-2x-4}{x+2} = \frac{-2(x+2)}{x+2} = -2$$

48. $$\frac{6-x}{x(x+2)} + \frac{x+2}{x^2} + \frac{8}{x^2(x+2)} = \frac{x(6-x)+(x+2)^2+8}{x^2(x+2)} = \frac{6x-x^2+x^2+4+8}{x^2(x+2)} = \frac{6(x+2)}{x^2(x+2)} = \frac{6}{x^2}$$

Simplifying a Complex Fraction **In Exercises 49–54, simplify the complex fraction.**

49. $\dfrac{\left(\dfrac{x}{2} - 1\right)}{(x-2)}$

50. $\dfrac{(x-4)}{\left(\dfrac{x}{4} - \dfrac{4}{x}\right)}$

51. $\dfrac{\left[\dfrac{x^2}{(x+1)^2}\right]}{\left[\dfrac{x}{(x+1)^3}\right]}$

52. $\dfrac{\left(\dfrac{x^2-1}{x}\right)}{\left[\dfrac{(x-1)^2}{x}\right]}$

53. $\dfrac{\left(\sqrt{x} - \dfrac{1}{2\sqrt{x}}\right)}{\sqrt{x}}$

54. $\dfrac{\left(\dfrac{t^2}{\sqrt{t^2+1}} - \sqrt{t^2+1}\right)}{t^2}$

Factoring an Expression **In Exercises 55–60, factor the expression by removing the common factor with the lesser exponent.**

55. $x^5 - 2x^{-2}$

56. $5x^5 + x^{-3}$

57. $x^2(x^2+1)^{-5} - (x^2+1)^{-4}$

58. $2x(x-5)^{-3} - 4x^2(x-5)^{-4}$

59. $2x^2(x-1)^{1/2} - 5(x-1)^{-1/2}$

60. $4x^3(2x-1)^{3/2} - 2x(2x-1)^{-1/2}$

Simplifying an Expression **In Exercises 61 and 62, simplify the expression.**

61. $\dfrac{3x^{1/3} - x^{-2/3}}{3x^{-2/3}}$

62. $\dfrac{-x^3(1-x^2)^{-1/2} - 2x(1-x^2)^{1/2}}{x^4}$

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Simplifying a Difference Quotient **In Exercises 63–66, simplify the difference quotient.**

63. $\dfrac{\left(\dfrac{1}{x+h} - \dfrac{1}{x}\right)}{h}$

64. $\dfrac{\left[\dfrac{1}{(x+h)^2} - \dfrac{1}{x^2}\right]}{h}$

65. $\dfrac{\left(\dfrac{1}{x+h-4} - \dfrac{1}{x-4}\right)}{h}$

66. $\dfrac{\left(\dfrac{x+h}{x+h+1} - \dfrac{x}{x+1}\right)}{h}$

Rewriting a Difference Quotient **In Exercises 67–72, rewrite the difference quotient by rationalizing the numerator.**

67. $\dfrac{\sqrt{x+2} - \sqrt{x}}{2}$

68. $\dfrac{\sqrt{z-3} - \sqrt{z}}{3}$

69. $\dfrac{\sqrt{t+3} - \sqrt{3}}{t}$

70. $\dfrac{\sqrt{x+5} - \sqrt{5}}{x}$

71. $\dfrac{\sqrt{x+h+1} - \sqrt{x+1}}{h}$

72. $\dfrac{\sqrt{x+h-2} - \sqrt{x-2}}{h}$

73. Refrigeration

After placing food (at room temperature) in a refrigerator, the time required for the food to cool depends on the amount of food, the air circulation in the refrigerator, the original temperature of the food, and the temperature of the refrigerator. The model that gives the temperature of food that has an original temperature of 75°F and is placed in a 40°F refrigerator is

$$T = 10\left(\frac{4t^2 + 16t + 75}{t^2 + 4t + 10}\right)$$

where T is the temperature (in degrees Fahrenheit) and t is the time (in hours).

(a) Complete the table.

t	0	2	4	6	8	10	12
T							

t	14	16	18	20	22
T					

(b) What value of T does the mathematical model appear to be approaching?

74. Rate A digital copier copies in color at a rate of 50 pages per minute.

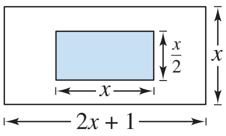
(a) Find the time required to copy one page.

(b) Find the time required to copy x pages.

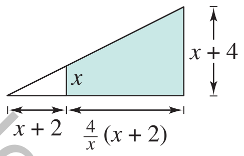
(c) Find the time required to copy 120 pages.

Probability In Exercises 75 and 76, consider an experiment in which a marble is tossed into a box whose base is shown in the figure. The probability that the marble will come to rest in the shaded portion of the box is equal to the ratio of the shaded area to the total area of the figure. Find the probability.

75.



76.



77. Interactive Money Management The table shows the numbers of U.S. households (in millions) banking online and paying bills online from 2005 through 2010. *(Source: Fiserv, Inc.)*

DATA

Spreadsheet at LarsonPrecalculus.com

Year	Banking	Paying Bills
2005	46.7	17.9
2006	58.6	26.3
2007	62.8	28.2
2008	67.0	30.0
2009	69.7	32.6
2010	72.5	36.4

Mathematical models for the data are

$$\text{Number banking online} = \frac{-33.74t + 121.8}{-0.40t + 1.0}$$

and

$$\text{Number paying bills online} = \frac{0.307t^2 - 6.54t + 24.6}{0.015t^2 - 0.28t + 1.0}$$

where t represents the year, with $t = 5$ corresponding to 2005.

(a) Using the models, create a table showing the numbers of households banking online and the numbers of households paying bills online for the given years.

(b) Compare the values given by the models with the actual data.

(c) Determine a model for the ratio of the number of households paying bills online to the number of households banking online.

(d) Use the model from part (c) to find the ratios for the given years. Interpret your results.

78. Finance The formula that approximates the annual interest rate r of a monthly installment loan is

$$r = \frac{24(NM - P)}{N} \div \left(P + \frac{NM}{12}\right)$$

where N is the total number of payments, M is the monthly payment, and P is the amount financed.

(a) Approximate the annual interest rate for a five-year car loan of \$28,000 that has monthly payments of \$525.

(b) Simplify the expression for the annual interest rate r, and then rework part (a).

79. Electrical Engineering The total resistance R_T (in ohms) of two resistors connected in parallel is given by

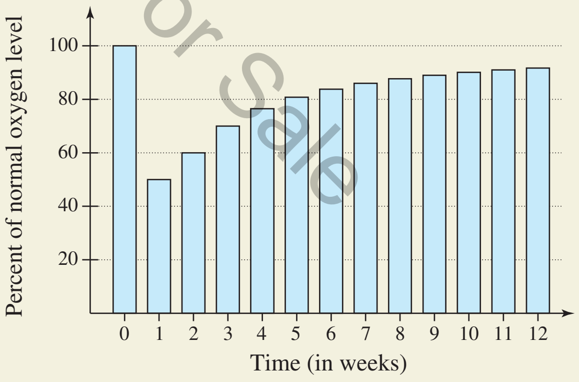
$$R_T = \frac{1}{\frac{1}{R_1} + \frac{1}{R_2}}$$

where R_1 and R_2 are the resistance values of the first and second resistors, respectively. Simplify the expression for the total resistance R_T.

80. HOW DO YOU SEE IT? The mathematical model

$$P = 100\left(\frac{t^2 - t + 1}{t^2 + 1}\right), \quad t \geq 0$$

gives the percent P of the normal level of oxygen in a pond, where t is the time (in weeks) after organic waste is dumped into the pond. The bar graph shows the situation. What conclusions can you draw from the bar graph?



Exploration

True or False? In Exercises 81 and 82, determine whether the statement is true or false. Justify your answer.

81. $\dfrac{x^{2n} - 1^{2n}}{x^n - 1^n} = x^n + 1^n$

82. $\dfrac{x^2 - 3x + 2}{x - 1} = x - 2$, for all values of x

A.5 Solving Equations

You can use linear equations in many real-life applications. For example, you can use linear equations in forensics to determine height from femur length. See Exercises 95 and 96 on page A57.

- **Identify different types of equations.**
- **Solve linear equations in one variable and rational equations that lead to linear equations.**
- **Solve quadratic equations by factoring, extracting square roots, completing the square, and using the Quadratic Formula.**
- **Solve polynomial equations of degree three or greater.**
- **Solve radical equations.**
- **Solve absolute value equations.**
- **Use common formulas to solve real-life problems.**

Equations and Solutions of Equations

An **equation** in x is a statement that two algebraic expressions are equal. For example,

$$3x - 5 = 7, \quad x^2 - x - 6 = 0, \quad \text{and} \quad \sqrt{2x} = 4$$

are equations. To **solve** an equation in x means to find all values of x for which the equation is true. Such values are **solutions.** For instance, $x = 4$ is a solution of the equation $3x - 5 = 7$ because $3(4) - 5 = 7$ is a true statement.

The solutions of an equation depend on the kinds of numbers being considered. For instance, in the set of rational numbers, $x^2 = 10$ has no solution because there is no rational number whose square is 10. However, in the set of real numbers, the equation has the two solutions $x = \sqrt{10}$ and $x = -\sqrt{10}$.

An equation that is true for *every* real number in the domain of the variable is called an **identity.** For example,

$$x^2 - 9 = (x + 3)(x - 3) \qquad \text{Identity}$$

is an identity because it is a true statement for any real value of x. The equation

$$\frac{x}{3x^2} = \frac{1}{3x} \qquad \text{Identity}$$

where $x \neq 0$, is an identity because it is true for any nonzero real value of x.

An equation that is true for just *some* (but not all) of the real numbers in the domain of the variable is called a **conditional equation.** For example, the equation

$$x^2 - 9 = 0 \qquad \text{Conditional equation}$$

is conditional because $x = 3$ and $x = -3$ are the only values in the domain that satisfy the equation.

A **contradiction** is an equation that is false for *every* real number in the domain of the variable. For example, the equation

$$2x - 4 = 2x + 1 \qquad \text{Contradiction}$$

is a contradiction because there are no real values of x for which the equation is true.

Linear and Rational Equations

Definition of Linear Equation in One Variable

A **linear equation in one variable** x is an equation that can be written in the standard form

$$ax + b = 0$$

where a and b are real numbers with $a \neq 0$.

Andrew Douglas/Masterfile

A linear equation has exactly one solution. To see this, consider the following steps. (Remember that $a \neq 0$.)

$ax + b = 0$ Write original equation.

$ax = -b$ Subtract b from each side.

$x = -\frac{b}{a}$ Divide each side by a.

To solve a conditional equation in x, isolate x on one side of the equation by a sequence of **equivalent equations,** each having the same solution(s) as the original equation. The operations that yield equivalent equations come from the properties of equality reviewed in Appendix A.1.

Generating Equivalent Equations

An equation can be transformed into an *equivalent equation* by one or more of the following steps.

	Given Equation	Equivalent Equation
1. Remove symbols of grouping, combine like terms, or simplify fractions on one or both sides of the equation.	$2x - x = 4$	$x = 4$
2. Add (or subtract) the same quantity to (from) *each* side of the equation.	$x + 1 = 6$	$x = 5$
3. Multiply (or divide) *each* side of the equation by the same *nonzero* quantity.	$2x = 6$	$x = 3$
4. Interchange the two sides of the equation.	$2 = x$	$x = 2$

The following example shows the steps for solving a linear equation in one variable x.

EXAMPLE 1 Solving a Linear Equation

a. $3x - 6 = 0$ Original equation

$3x = 6$ Add 6 to each side.

$x = 2$ Divide each side by 3.

b. $5x + 4 = 3x - 8$ Original equation

$2x + 4 = -8$ Subtract $3x$ from each side.

$2x = -12$ Subtract 4 from each side.

$x = -6$ Divide each side by 2.

REMARK After solving an equation, you should check each solution in the original equation. For instance, you can check the solution of Example 1(a) as follows.

$3x - 6 = 0$ Write original equation.

$3(2) - 6 \overset{?}{=} 0$ Substitute 2 for x.

$0 = 0$ Solution checks. ✓

Try checking the solution of Example 1(b).

✓ **Checkpoint** *Audio-video solution in English & Spanish at LarsonPrecalculus.com.*

Solve each equation.

a. $7 - 2x = 15$

b. $7x - 9 = 5x + 7$

A **rational equation** is an equation that involves one or more fractional expressions. To solve a rational equation, find the least common denominator (LCD) of all terms and multiply every term by the LCD. This process will clear the original equation of fractions and produce a simpler equation to work with.

REMARK An equation with a *single fraction* on each side can be cleared of denominators by **cross multiplying**. To do this, multiply the left numerator by the right denominator and the right numerator by the left denominator as follows.

$$\begin{aligned} \frac{a}{b} &= \frac{c}{d} && \text{Original equation} \\ ad &= cb && \text{Cross multiply.} \end{aligned}$$

EXAMPLE 2 Solving a Rational Equation

Solve $\frac{x}{3} + \frac{3x}{4} = 2$.

Solution

$$\begin{aligned} \frac{x}{3} + \frac{3x}{4} &= 2 && \text{Original equation} \\ (12)\frac{x}{3} + (12)\frac{3x}{4} &= (12)2 && \text{Multiply each term by the LCD.} \\ 4x + 9x &= 24 && \text{Simplify.} \\ 13x &= 24 && \text{Combine like terms.} \\ x &= \frac{24}{13} && \text{Divide each side by 13.} \end{aligned}$$

The solution is $x = \frac{24}{13}$. Check this in the original equation.

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Solve $\frac{4x}{9} - \frac{1}{3} = x + \frac{5}{3}$.

When multiplying or dividing an equation by a *variable expression,* it is possible to introduce an **extraneous solution** that does not satisfy the original equation.

EXAMPLE 3 An Equation with an Extraneous Solution

Solve $\frac{1}{x-2} = \frac{3}{x+2} - \frac{6x}{x^2-4}$.

Solution The LCD is $x^2 - 4 = (x + 2)(x - 2)$. Multiply each term by this LCD.

$$\begin{aligned} \frac{1}{x-2}(x+2)(x-2) &= \frac{3}{x+2}(x+2)(x-2) - \frac{6x}{x^2-4}(x+2)(x-2) \\ x + 2 &= 3(x-2) - 6x, \quad x \neq \pm 2 \\ x + 2 &= 3x - 6 - 6x \\ x + 2 &= -3x - 6 \\ 4x &= -8 \\ x &= -2 && \text{Extraneous solution} \end{aligned}$$

In the original equation, $x = -2$ yields a denominator of zero. So, $x = -2$ is an extraneous solution, and the original equation has *no solution.*

REMARK Recall that the least common denominator of two or more fractions consists of the product of all prime factors in the denominators, with each factor given the highest power of its occurrence in any denominator. For instance, in Example 3, by factoring each denominator you can determine that the LCD is $(x + 2)(x - 2)$.

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Solve $\frac{3x}{x-4} = 5 + \frac{12}{x-4}$.

Quadratic Equations

A **quadratic equation** in x is an equation that can be written in the general form

$$ax^2 + bx + c = 0$$

where a, b, and c are real numbers with $a \neq 0$. A quadratic equation in x is also called a **second-degree polynomial equation** in x.

You should be familiar with the following four methods of solving quadratic equations.

Solving a Quadratic Equation

Factoring

If $ab = 0$, then $a = 0$ or $b = 0$.

Example:

$$\begin{aligned} x^2 - x - 6 &= 0 \\ (x - 3)(x + 2) &= 0 \\ x - 3 &= 0 \quad \Rightarrow \quad x = 3 \\ x + 2 &= 0 \quad \Rightarrow \quad x = -2 \end{aligned}$$

REMARK The Square Root Principle is also referred to as *extracting square roots.*

Square Root Principle

If $u^2 = c$, where $c > 0$, then $u = \pm\sqrt{c}$.

Example:

$$\begin{aligned} (x + 3)^2 &= 16 \\ x + 3 &= \pm 4 \\ x &= -3 \pm 4 \\ x &= 1 \quad \text{or} \quad x = -7 \end{aligned}$$

Completing the Square

If $x^2 + bx = c$, then

$$\begin{aligned} x^2 + bx + \left(\frac{b}{2}\right)^2 &= c + \left(\frac{b}{2}\right)^2 && \text{Add } \left(\frac{b}{2}\right)^2 \text{ to each side.} \\ \left(x + \frac{b}{2}\right)^2 &= c + \frac{b^2}{4}. \end{aligned}$$

Example:

$$\begin{aligned} x^2 + 6x &= 5 \\ x^2 + 6x + 3^2 &= 5 + 3^2 && \text{Add } \left(\frac{6}{2}\right)^2 \text{ to each side.} \\ (x + 3)^2 &= 14 \\ x + 3 &= \pm\sqrt{14} \\ x &= -3 \pm \sqrt{14} \end{aligned}$$

REMARK You can solve every quadratic equation by completing the square or using the Quadratic Formula.

Quadratic Formula

If $ax^2 + bx + c = 0$, then $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$.

Example:

$$\begin{aligned} 2x^2 + 3x - 1 &= 0 \\ x &= \frac{-3 \pm \sqrt{3^2 - 4(2)(-1)}}{2(2)} \\ &= \frac{-3 \pm \sqrt{17}}{4} \end{aligned}$$

EXAMPLE 4 Solving a Quadratic Equation by Factoring

a. $2x^2 + 9x + 7 = 3$ Original equation

$2x^2 + 9x + 4 = 0$ Write in general form.

$(2x + 1)(x + 4) = 0$ Factor.

$2x + 1 = 0 \Rightarrow x = -\frac{1}{2}$ Set 1st factor equal to 0.

$x + 4 = 0 \Rightarrow x = -4$ Set 2nd factor equal to 0.

The solutions are $x = -\frac{1}{2}$ and $x = -4$. Check these in the original equation.

b. $6x^2 - 3x = 0$ Original equation

$3x(2x - 1) = 0$ Factor.

$3x = 0 \Rightarrow x = 0$ Set 1st factor equal to 0.

$2x - 1 = 0 \Rightarrow x = \frac{1}{2}$ Set 2nd factor equal to 0.

The solutions are $x = 0$ and $x = \frac{1}{2}$. Check these in the original equation.

✓ ***Checkpoint*** *Audio-video solution in English & Spanish at LarsonPrecalculus.com.*

Solve $2x^2 - 3x + 1 = 6$ by factoring. ■

Note that the method of solution in Example 4 is based on the Zero-Factor Property from Appendix A.1. This property applies *only* to equations written in general form (in which the right side of the equation is zero). So, all terms must be collected on one side *before* factoring. For instance, in the equation $(x - 5)(x + 2) = 8$, it is *incorrect* to set each factor equal to 8. Try to solve this equation correctly.

EXAMPLE 5 Extracting Square Roots

Solve each equation by extracting square roots.

a. $4x^2 = 12$

b. $(x - 3)^2 = 7$

Solution

a. $4x^2 = 12$ Write original equation.

$x^2 = 3$ Divide each side by 4.

$x = \pm\sqrt{3}$ Extract square roots.

The solutions are $x = \sqrt{3}$ and $x = -\sqrt{3}$. Check these in the original equation.

b. $(x - 3)^2 = 7$ Write original equation.

$x - 3 = \pm\sqrt{7}$ Extract square roots.

$x = 3 \pm \sqrt{7}$ Add 3 to each side.

The solutions are $x = 3 \pm \sqrt{7}$. Check these in the original equation.

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Solve each equation by extracting square roots.

a. $3x^2 = 36$

b. $(x - 1)^2 = 10$ ■

When solving quadratic equations by completing the square, you must add $(b/2)^2$ to *each side* in order to maintain equality. When the leading coefficient is *not* 1, you must divide each side of the equation by the leading coefficient *before* completing the square, as shown in Example 7.

EXAMPLE 6 Completing the Square: Leading Coefficient Is 1

Solve $x^2 + 2x - 6 = 0$ by completing the square.

Solution

$$\begin{aligned} x^2 + 2x - 6 &= 0 && \text{Write original equation.} \\ x^2 + 2x &= 6 && \text{Add 6 to each side.} \\ x^2 + 2x + 1^2 &= 6 + 1^2 && \text{Add } 1^2 \text{ to each side.} \end{aligned}$$

$(\text{half of } 2)^2$

$$\begin{aligned} (x + 1)^2 &= 7 && \text{Simplify.} \\ x + 1 &= \pm\sqrt{7} && \text{Extract square roots.} \\ x &= -1 \pm \sqrt{7} && \text{Subtract 1 from each side.} \end{aligned}$$

The solutions are

$$x = -1 \pm \sqrt{7}.$$

Check these in the original equation.

✓ **Checkpoint** *Audio-video solution in English & Spanish at LarsonPrecalculus.com.*

Solve $x^2 - 4x - 1 = 0$ by completing the square.

EXAMPLE 7 Completing the Square: Leading Coefficient Is Not 1

Solve $3x^2 - 4x - 5 = 0$ by completing the square.

Solution

$$\begin{aligned} 3x^2 - 4x - 5 &= 0 && \text{Write original equation.} \\ 3x^2 - 4x &= 5 && \text{Add 5 to each side.} \\ x^2 - \frac{4}{3}x &= \frac{5}{3} && \text{Divide each side by 3.} \\ x^2 - \frac{4}{3}x + \left(-\frac{2}{3}\right)^2 &= \frac{5}{3} + \left(-\frac{2}{3}\right)^2 && \text{Add } \left(-\tfrac{2}{3}\right)^2 \text{ to each side.} \end{aligned}$$

$\left(\text{half of } -\frac{4}{3}\right)^2$

$$\begin{aligned} \left(x - \frac{2}{3}\right)^2 &= \frac{19}{9} && \text{Simplify.} \\ x - \frac{2}{3} &= \pm\frac{\sqrt{19}}{3} && \text{Extract square roots.} \\ x &= \frac{2}{3} \pm \frac{\sqrt{19}}{3} && \text{Add } \tfrac{2}{3} \text{ to each side.} \end{aligned}$$

✓ **Checkpoint** *Audio-video solution in English & Spanish at LarsonPrecalculus.com.*

Solve $3x^2 - 10x - 2 = 0$ by completing the square.

EXAMPLE 8 The Quadratic Formula: Two Distinct Solutions

Use the Quadratic Formula to solve $x^2 + 3x = 9$.

Solution

$$\begin{aligned}
x^2 + 3x &= 9 && \text{Write original equation.}\\
x^2 + 3x - 9 &= 0 && \text{Write in general form.}\\
x &= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} && \text{Quadratic Formula}\\
x &= \frac{-3 \pm \sqrt{(3)^2 - 4(1)(-9)}}{2(1)} && \text{Substitute } a = 1,\ b = 3, \text{ and } c = -9.\\
x &= \frac{-3 \pm \sqrt{45}}{2} && \text{Simplify.}\\
x &= \frac{-3 \pm 3\sqrt{5}}{2} && \text{Simplify.}
\end{aligned}$$

REMARK When using the Quadratic Formula, remember that *before* applying the formula, you must first write the quadratic equation in general form.

The two solutions are

$$x = \frac{-3 + 3\sqrt{5}}{2} \quad \text{and} \quad x = \frac{-3 - 3\sqrt{5}}{2}.$$

Check these in the original equation.

✓ ***Checkpoint*** *Audio-video solution in English & Spanish at LarsonPrecalculus.com.*

Use the Quadratic Formula to solve $3x^2 + 2x - 10 = 0$.

EXAMPLE 9 The Quadratic Formula: One Solution

Use the Quadratic Formula to solve $8x^2 - 24x + 18 = 0$.

Solution

$$\begin{aligned}
8x^2 - 24x + 18 &= 0 && \text{Write original equation.}\\
4x^2 - 12x + 9 &= 0 && \text{Divide out common factor of 2.}\\
x &= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} && \text{Quadratic Formula}\\
x &= \frac{-(-12) \pm \sqrt{(-12)^2 - 4(4)(9)}}{2(4)} && \text{Substitute } a = 4,\ b = -12, \text{ and } c = 9.\\
x &= \frac{12 \pm \sqrt{0}}{8} = \frac{3}{2} && \text{Simplify.}
\end{aligned}$$

This quadratic equation has only one solution:

$$x = \tfrac{3}{2}.$$

Check this in the original equation.

✓ ***Checkpoint*** *Audio-video solution in English & Spanish at LarsonPrecalculus.com.*

Use the Quadratic Formula to solve $18x^2 - 48x + 32 = 0$.

Note that you could have solved Example 9 without first dividing out a common factor of 2. Substituting $a = 8$, $b = -24$, and $c = 18$ into the Quadratic Formula produces the same result.

Polynomial Equations of Higher Degree

The methods used to solve quadratic equations can sometimes be extended to solve polynomial equations of higher degrees.

REMARK A common mistake in solving an equation such as that in Example 10 is to divide each side of the equation by the variable factor x^2. This loses the solution $x = 0$. When solving an equation, always write the equation in general form, then factor the equation and set each factor equal to zero. Do not divide each side of an equation by a variable factor in an attempt to simplify the equation.

EXAMPLE 10 Solving a Polynomial Equation by Factoring

Solve $3x^4 = 48x^2$.

Solution First write the polynomial equation in general form with zero on one side. Then factor the other side, set each factor equal to zero, and solve.

$$\begin{aligned} 3x^4 &= 48x^2 && \text{Write original equation.} \\ 3x^4 - 48x^2 &= 0 && \text{Write in general form.} \\ 3x^2(x^2 - 16) &= 0 && \text{Factor out common factor.} \\ 3x^2(x + 4)(x - 4) &= 0 && \text{Write in factored form.} \\ 3x^2 = 0 &\Rightarrow x = 0 && \text{Set 1st factor equal to 0.} \\ x + 4 = 0 &\Rightarrow x = -4 && \text{Set 2nd factor equal to 0.} \\ x - 4 = 0 &\Rightarrow x = 4 && \text{Set 3rd factor equal to 0.} \end{aligned}$$

You can check these solutions by substituting in the original equation, as follows.

Check

$$\begin{aligned} 3(0)^4 &= 48(0)^2 && \text{0 checks. ✓} \\ 3(-4)^4 &= 48(-4)^2 && \text{−4 checks. ✓} \\ 3(4)^4 &= 48(4)^2 && \text{4 checks. ✓} \end{aligned}$$

So, the solutions are

$$x = 0, \quad x = -4, \quad \text{and} \quad x = 4.$$

✓ ***Checkpoint*** *Audio-video solution in English & Spanish at LarsonPrecalculus.com.*

Solve $9x^4 - 12x^2 = 0$.

EXAMPLE 11 Solving a Polynomial Equation by Factoring

Solve $x^3 - 3x^2 - 3x + 9 = 0$.

Solution

$$\begin{aligned} x^3 - 3x^2 - 3x + 9 &= 0 && \text{Write original equation.} \\ x^2(x - 3) - 3(x - 3) &= 0 && \text{Factor by grouping.} \\ (x - 3)(x^2 - 3) &= 0 && \text{Distributive Property} \\ x - 3 = 0 &\Rightarrow x = 3 && \text{Set 1st factor equal to 0.} \\ x^2 - 3 = 0 &\Rightarrow x = \pm\sqrt{3} && \text{Set 2nd factor equal to 0.} \end{aligned}$$

The solutions are $x = 3$, $x = \sqrt{3}$, and $x = -\sqrt{3}$. Check these in the original equation.

✓ ***Checkpoint*** *Audio-video solution in English & Spanish at LarsonPrecalculus.com.*

Solve each equation.

a. $x^3 - 5x^2 - 2x + 10 = 0$

b. $6x^3 - 27x^2 - 54x = 0$

Radical Equations

A **radical equation** is an equation that involves one or more radical expressions.

REMARK When squaring each side of an equation or raising each side of an equation to a rational power, it is possible to introduce extraneous solutions. In such cases, checking your solutions is crucial.

EXAMPLE 12 Solving Radical Equations

a.

$$\begin{aligned}
\sqrt{2x+7} - x &= 2 && \text{Original equation}\\
\sqrt{2x+7} &= x + 2 && \text{Isolate radical.}\\
2x + 7 &= x^2 + 4x + 4 && \text{Square each side.}\\
0 &= x^2 + 2x - 3 && \text{Write in general form.}\\
0 &= (x+3)(x-1) && \text{Factor.}\\
x + 3 = 0 \;&\Rightarrow\; x = -3 && \text{Set 1st factor equal to 0.}\\
x - 1 = 0 \;&\Rightarrow\; x = 1 && \text{Set 2nd factor equal to 0.}
\end{aligned}$$

By checking these values, you can determine that the only solution is $x = 1$.

REMARK When an equation contains two radicals, it may not be possible to isolate both. In such cases, you may have to raise each side of the equation to a power at *two* different stages in the solution, as shown in Example 12(b).

b.

$$\begin{aligned}
\sqrt{2x-5} - \sqrt{x-3} &= 1 && \text{Original equation}\\
\sqrt{2x-5} &= \sqrt{x-3} + 1 && \text{Isolate } \sqrt{2x-5}.\\
2x - 5 &= x - 3 + 2\sqrt{x-3} + 1 && \text{Square each side.}\\
2x - 5 &= x - 2 + 2\sqrt{x-3} && \text{Combine like terms.}\\
x - 3 &= 2\sqrt{x-3} && \text{Isolate } 2\sqrt{x-3}.\\
x^2 - 6x + 9 &= 4(x-3) && \text{Square each side.}\\
x^2 - 10x + 21 &= 0 && \text{Write in general form.}\\
(x-3)(x-7) &= 0 && \text{Factor.}\\
x - 3 = 0 \;&\Rightarrow\; x = 3 && \text{Set 1st factor equal to 0.}\\
x - 7 = 0 \;&\Rightarrow\; x = 7 && \text{Set 2nd factor equal to 0.}
\end{aligned}$$

The solutions are $x = 3$ and $x = 7$. Check these in the original equation.

✓ ***Checkpoint*** *Audio-video solution in English & Spanish at LarsonPrecalculus.com.*

Solve $-\sqrt{40 - 9x} + 2 = x$.

EXAMPLE 13 Solving an Equation Involving a Rational Exponent

Solve $(x - 4)^{2/3} = 25$.

Solution

$$\begin{aligned}
(x-4)^{2/3} &= 25 && \text{Write original equation.}\\
\sqrt[3]{(x-4)^2} &= 25 && \text{Rewrite in radical form.}\\
(x-4)^2 &= 15{,}625 && \text{Cube each side.}\\
x - 4 &= \pm 125 && \text{Extract square roots.}\\
x &= 129, \quad x = -121 && \text{Add 4 to each side.}
\end{aligned}$$

✓ ***Checkpoint*** *Audio-video solution in English & Spanish at LarsonPrecalculus.com.*

Solve $(x - 5)^{2/3} = 16$.

Absolute Value Equations

An **absolute value equation** is an equation that involves one or more absolute value expressions. To solve an absolute value equation, remember that the expression inside the absolute value bars can be positive or negative. This results in *two* separate equations, each of which must be solved. For instance, the equation

$$|x-2|=3$$

results in the two equations

$$x-2=3 \quad \text{and} \quad -(x-2)=3$$

which implies that the equation has two solutions: $x=5$ and $x=-1$.

EXAMPLE 14 Solving an Absolute Value Equation

Solve $|x^2-3x|=-4x+6$.

Solution Because the variable expression inside the absolute value bars can be positive or negative, you must solve the following two equations.

First Equation

$$x^2-3x=-4x+6$$ Use positive expression.

$$x^2+x-6=0$$ Write in general form.

$$(x+3)(x-2)=0$$ Factor.

$$x+3=0 \Rightarrow x=-3$$ Set 1st factor equal to 0.

$$x-2=0 \Rightarrow x=2$$ Set 2nd factor equal to 0.

Second Equation

$$-(x^2-3x)=-4x+6$$ Use negative expression.

$$x^2-7x+6=0$$ Write in general form.

$$(x-1)(x-6)=0$$ Factor.

$$x-1=0 \Rightarrow x=1$$ Set 1st factor equal to 0.

$$x-6=0 \Rightarrow x=6$$ Set 2nd factor equal to 0.

Check

$$|(-3)^2-3(-3)| \stackrel{?}{=} -4(-3)+6$$ Substitute -3 for x.

$$18=18$$ -3 checks. ✓

$$|(2)^2-3(2)| \stackrel{?}{=} -4(2)+6$$ Substitute 2 for x.

$$2 \neq -2$$ 2 does not check.

$$|(1)^2-3(1)| \stackrel{?}{=} -4(1)+6$$ Substitute 1 for x.

$$2=2$$ 1 checks. ✓

$$|(6)^2-3(6)| \stackrel{?}{=} -4(6)+6$$ Substitute 6 for x.

$$18 \neq -18$$ 6 does not check.

The solutions are $x=-3$ and $x=1$.

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Solve $|x^2+4x|=5x+12$.

Common Formulas

The following geometric formulas are used at various times throughout this course. For your convenience, some of these formulas along with several others are also provided on the inside cover of this text.

Common Formulas for Area *A*, Perimeter *P*, Circumference *C*, and Volume *V*

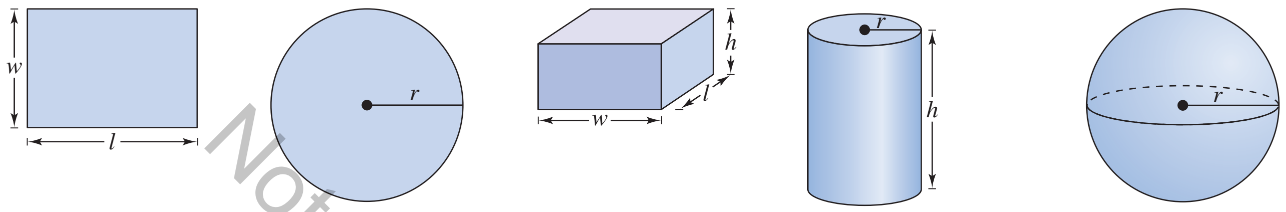
Rectangle
$A = lw$
$P = 2l + 2w$

Circle
$A = \pi r^2$
$C = 2\pi r$

Rectangular Solid
$V = lwh$

Circular Cylinder
$V = \pi r^2 h$

Sphere
$V = \frac{4}{3}\pi r^3$



EXAMPLE 15 Using a Geometric Formula

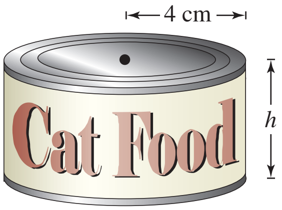


Figure A.10

The cylindrical can shown in Figure A.10 has a volume of 200 cubic centimeters (cm^3). Find the height of the can.

Solution The formula for the *volume of a cylinder* is $V = \pi r^2 h$. To find the height of the can, solve for *h*. Then, using $V = 200$ and $r = 4$, find the height.

$$V = \pi r^2 h \quad \Rightarrow \quad h = \frac{V}{\pi r^2} = \frac{200}{\pi(4)^2} = \frac{200}{16\pi} \approx 3.98$$

So, the height of the can is about 3.98 centimeters.

REMARK You can use unit analysis to check that the answer in Example 15 is reasonable.

$$\frac{200 \text{ cm}^3}{16\pi \text{ cm}^2} = \frac{200 \cancel{\text{cm}} \cdot \cancel{\text{cm}} \cdot \text{cm}}{16\pi \cancel{\text{cm}} \cdot \cancel{\text{cm}}}$$
$$= \frac{200}{16\pi} \text{ cm}$$
$$\approx 3.98 \text{ cm}$$

✓ ***Checkpoint*** *Audio-video solution in English & Spanish at LarsonPrecalculus.com.*

A cylindrical container has a volume of 84 cubic inches and a radius of 3 inches. Find the height of the container.

Summarize (Appendix A.5)

1. State the definition of an identity, a conditional equation, and a contradiction *(page A45)*.
2. State the definition of a linear equation *(page A45)*. For examples of solving linear equations and rational equations that lead to linear equations, see Examples 1–3.
3. List the four methods of solving quadratic equations discussed in this section *(page A48)*. For examples of solving quadratic equations, see Examples 4–9.
4. Describe how to solve a polynomial equation by factoring *(page A52)*. For examples of solving polynomial equations by factoring, see Examples 10 and 11.
5. Describe how to solve a radical equation *(page A53)*. For an example of solving radical equations, see Example 12.
6. Describe how to solve an absolute value equation *(page A54)*. For an example of solving an absolute value equation, see Example 14.
7. Describe real-life problems that can be solved using common geometric formulas *(page A55)*. For an example that uses a volume formula, see Example 15.

A.5 Exercises

See **CalcChat.com** for tutorial help and worked-out solutions to odd-numbered exercises.

Vocabulary: Fill in the blanks.

1. An ________ is a statement that equates two algebraic expressions.

2. A linear equation in one variable x is an equation that can be written in the standard form ________.

3. An ________ solution is a solution that does not satisfy the original equation.

4. Four methods that can be used to solve a quadratic equation are ________, extracting ________ ________, ________ the ________, and the ________ ________.

Skills and Applications

Solving a Linear Equation

In Exercises 5–12, solve the equation and check your solution. (If not possible, explain why.)

5. $x + 11 = 15$

6. $7 - x = 19$

7. $7 - 2x = 25$

8. $3x - 5 = 2x + 7$

9. $4y + 2 - 5y = 7 - 6y$

10. $0.25x + 0.75(10 - x) = 3$

11. $x - 3(2x + 3) = 8 - 5x$

12. $9x - 10 = 5x + 2(2x - 5)$

Solving a Rational Equation

In Exercises 13–24, solve the equation and check your solution. (If not possible, explain why.)

13. $\frac{3x}{8} - \frac{4x}{3} = 4$

14. $\frac{5x}{4} + \frac{1}{2} = x - \frac{1}{2}$

15. $\frac{5x - 4}{5x + 4} = \frac{2}{3}$

16. $\frac{10x + 3}{5x + 6} = \frac{1}{2}$

17. $10 - \frac{13}{x} = 4 + \frac{5}{x}$

18. $\frac{1}{x} + \frac{2}{x - 5} = 0$

19. $\frac{x}{x + 4} + \frac{4}{x + 4} + 2 = 0$

20. $\frac{7}{2x + 1} - \frac{8x}{2x - 1} = -4$

21. $\frac{2}{(x - 4)(x - 2)} = \frac{1}{x - 4} + \frac{2}{x - 2}$

22. $\frac{4}{x - 1} + \frac{6}{3x + 1} = \frac{15}{3x + 1}$

23. $\frac{1}{x - 3} + \frac{1}{x + 3} = \frac{10}{x^2 - 9}$

24. $\frac{1}{x - 2} + \frac{3}{x + 3} = \frac{4}{x^2 + x - 6}$

Solving a Quadratic Equation by Factoring

In Exercises 25–34, solve the quadratic equation by factoring.

25. $6x^2 + 3x = 0$

26. $9x^2 - 1 = 0$

27. $x^2 - 2x - 8 = 0$

28. $x^2 - 10x + 9 = 0$

29. $x^2 + 10x + 25 = 0$

30. $4x^2 + 12x + 9 = 0$

31. $x^2 + 4x = 12$

32. $-x^2 + 8x = 12$

33. $\frac{3}{4}x^2 + 8x + 20 = 0$

34. $\frac{1}{8}x^2 - x - 16 = 0$

Extracting Square Roots

In Exercises 35–42, solve the equation by extracting square roots. When a solution is irrational, list both the exact solution *and* its approximation rounded to two decimal places.

35. $x^2 = 49$

36. $x^2 = 32$

37. $3x^2 = 81$

38. $9x^2 = 36$

39. $(x - 12)^2 = 16$

40. $(x + 9)^2 = 24$

41. $(2x - 1)^2 = 18$

42. $(x - 7)^2 = (x + 3)^2$

Completing the Square

In Exercises 43–50, solve the quadratic equation by completing the square.

43. $x^2 + 4x - 32 = 0$

44. $x^2 - 2x - 3 = 0$

45. $x^2 + 6x + 2 = 0$

46. $x^2 + 8x + 14 = 0$

47. $9x^2 - 18x = -3$

48. $7 + 2x - x^2 = 0$

49. $2x^2 + 5x - 8 = 0$

50. $3x^2 - 4x - 7 = 0$

Using the Quadratic Formula

In Exercises 51–64, use the Quadratic Formula to solve the equation.

51. $2x^2 + x - 1 = 0$

52. $2x^2 - x - 1 = 0$

53. $2 + 2x - x^2 = 0$

54. $x^2 - 10x + 22 = 0$

55. $2x^2 - 3x - 4 = 0$

56. $3x + x^2 - 1 = 0$

57. $12x - 9x^2 = -3$

58. $9x^2 - 37 = 6x$

59. $9x^2 + 30x + 25 = 0$

60. $28x - 49x^2 = 4$

61. $8t = 5 + 2t^2$

62. $25h^2 + 80h + 61 = 0$

63. $(y - 5)^2 = 2y$

64. $(z + 6)^2 = -2z$

Choosing a Method **In Exercises 65–72, solve the equation using any convenient method.**

65. $x^2 - 2x - 1 = 0$

66. $11x^2 + 33x = 0$

67. $(x + 3)^2 = 81$

68. $x^2 - 14x + 49 = 0$

69. $x^2 - x - \frac{11}{4} = 0$

70. $x^2 + 3x - \frac{3}{4} = 0$

71. $(x + 1)^2 = x^2$

72. $3x + 4 = 2x^2 - 7$

Solving a Polynomial Equation **In Exercises 73–76, solve the equation. Check your solutions.**

73. $6x^4 - 14x^2 = 0$

74. $36x^3 - 100x = 0$

75. $5x^3 + 30x^2 + 45x = 0$

76. $x^3 - 3x^2 - x = -3$

Solving a Radical Equation **In Exercises 77–84, solve the equation. Check your solutions.**

77. $\sqrt{3x} - 12 = 0$

78. $\sqrt{x - 10} - 4 = 0$

79. $\sqrt[3]{2x + 5} + 3 = 0$

80. $\sqrt[3]{3x + 1} - 5 = 0$

81. $-\sqrt{26 - 11x} + 4 = x$

82. $x + \sqrt{31 - 9x} = 5$

83. $\sqrt{x} - \sqrt{x - 5} = 1$

84. $2\sqrt{x + 1} - \sqrt{2x + 3} = 1$

Solving an Equation Involving a Rational Exponent **In Exercises 85–88, solve the equation. Check your solutions.**

85. $(x - 5)^{3/2} = 8$

86. $(x + 2)^{2/3} = 9$

87. $(x^2 - 5)^{3/2} = 27$

88. $(x^2 - x - 22)^{3/2} = 27$

Solving an Absolute Value Equation **In Exercises 89–92, solve the equation. Check your solutions.**

89. $|2x - 5| = 11$

90. $|3x + 2| = 7$

91. $|x^2 + 6x| = 3x + 18$

92. $|x - 15| = x^2 - 15x$

93. **Volume of a Billiard Ball** A billiard ball has a volume of 5.96 cubic inches. Find the radius of a billiard ball.

94. **Length of a Tank** The diameter of a cylindrical propane gas tank is 4 feet. The total volume of the tank is 603.2 cubic feet. Find the length of the tank.

18percentgrey/Shutterstock.com

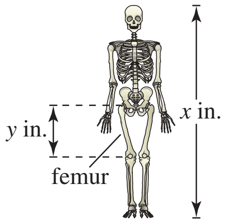
Forensics

In Exercises 95 and 96, use the following information. The relationship between the length of an adult's femur (thigh bone) and the height of the adult can be approximated by the linear equations

$y = 0.432x - 10.44$ Female

$y = 0.449x - 12.15$ Male

where y is the length of the femur in inches and x is the height of the adult in inches (see figure).



95. A crime scene investigator discovers a femur belonging to an adult human female. The bone is 18 inches long. Estimate the height of the female.

96. Officials search a forest for a missing man who is 6 feet 2 inches tall. They find an adult male femur that is 21 inches long. Is it possible that the femur belongs to the missing man?

Exploration

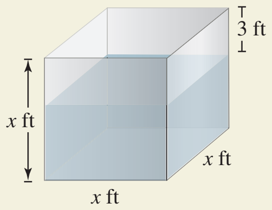
True or False? **In Exercises 97–99, determine whether the statement is true or false. Justify your answer.**

97. An equation can never have more than one extraneous solution.

98. The equation $2(x - 3) + 1 = 2x - 5$ has no solution.

99. The equation $\sqrt{x + 10} - \sqrt{x - 10} = 0$ has no solution.

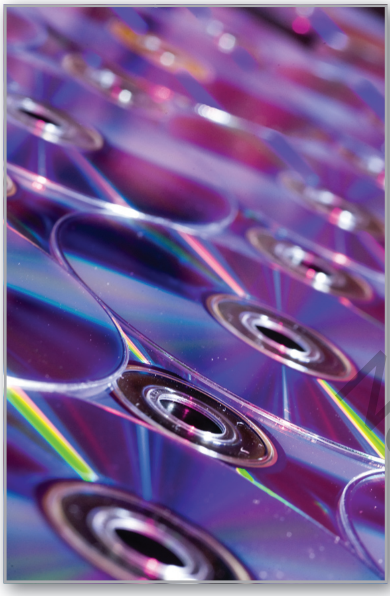
100. **HOW DO YOU SEE IT?** The figure shows a glass cube partially filled with water.



(a) What does the expression $x^2(x - 3)$ represent?

(b) Given $x^2(x - 3) = 320$, explain how you can find the capacity of the cube.

A.6 Linear Inequalities in One Variable



You can use inequalities to model and solve real-life problems. For instance, in Exercise 114 on page A66, you will use an absolute value inequality to analyze the cover layer thickness of a Blu-ray Disc™.

- **Represent solutions of linear inequalities in one variable.**
- **Use properties of inequalities to create equivalent inequalities.**
- **Solve linear inequalities in one variable.**
- **Solve absolute value inequalities.**
- **Use inequalities to model and solve real-life problems.**

Introduction

Simple inequalities were discussed in Appendix A.1. There, you used the inequality symbols $<$, $\leq$, $>$, and $\geq$ to compare two numbers and to denote subsets of real numbers. For instance, the simple inequality

$$x \geq 3$$

denotes all real numbers x that are greater than or equal to 3.

Now, you will expand your work with inequalities to include more involved statements such as

$$5x - 7 < 3x + 9 \quad \text{and} \quad -3 \leq 6x - 1 < 3.$$

As with an equation, you **solve an inequality** in the variable x by finding all values of x for which the inequality is true. Such values are **solutions** and are said to **satisfy** the inequality. The set of all real numbers that are solutions of an inequality is the **solution set** of the inequality. For instance, the solution set of

$$x + 1 < 4$$

is all real numbers that are less than 3.

The set of all points on the real number line that represents the solution set is the **graph of the inequality.** Graphs of many types of inequalities consist of intervals on the real number line. See Appendix A.1 to review the nine basic types of intervals on the real number line. Note that each type of interval can be classified as *bounded* or *unbounded.*

EXAMPLE 1 Intervals and Inequalities

Write an inequality to represent each interval. Then state whether the interval is bounded or unbounded.

a. $(-3, 5]$ **b.** $(-3, \infty)$

c. $[0, 2]$ **d.** $(-\infty, \infty)$

Solution

a. $(-3, 5]$ corresponds to $-3 < x \leq 5$. Bounded

b. $(-3, \infty)$ corresponds to $x > -3$. Unbounded

c. $[0, 2]$ corresponds to $0 \leq x \leq 2$. Bounded

d. $(-\infty, \infty)$ corresponds to $-\infty < x < \infty$. Unbounded

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Write an inequality to represent each interval. Then state whether the interval is bounded or unbounded.

a. $[-1, 3]$ **b.** $(-1, 6)$

c. $(-\infty, 4)$ **d.** $[0, \infty)$

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Properties of Inequalities

The procedures for solving linear inequalities in one variable are much like those for solving linear equations. To isolate the variable, you can make use of the **properties of inequalities.** These properties are similar to the properties of equality, but there are two important exceptions. When each side of an inequality is multiplied or divided by a negative number, the direction of the inequality symbol must be reversed. Here is an example.

$$-2 < 5 \quad \text{Original inequality}$$

$$(-3)(-2) > (-3)(5) \quad \text{Multiply each side by } -3 \text{ and reverse inequality symbol.}$$

$$6 > -15 \quad \text{Simplify.}$$

Notice that when you do not reverse the inequality symbol in the example above, you obtain the false statement

$$6 < -15. \quad \text{False statement}$$

Two inequalities that have the same solution set are **equivalent.** For instance, the inequalities

$$x + 2 < 5$$

and

$$x < 3$$

are equivalent. To obtain the second inequality from the first, you can subtract 2 from each side of the inequality. The following list describes the operations that can be used to create equivalent inequalities.

Properties of Inequalities

Let a, b, c, and d be real numbers.

1. Transitive Property

$$a < b \text{ and } b < c \Rightarrow a < c$$

2. Addition of Inequalities

$$a < b \text{ and } c < d \Rightarrow a + c < b + d$$

3. Addition of a Constant

$$a < b \Rightarrow a + c < b + c$$

4. Multiplication by a Constant

$$\text{For } c > 0,\ a < b \Rightarrow ac < bc$$

$$\text{For } c < 0,\ a < b \Rightarrow ac > bc \quad \text{Reverse the inequality symbol.}$$

Each of the properties above is true when the symbol $<$ is replaced by $\le$ and the symbol $>$ is replaced by $\ge$. For instance, another form of the multiplication property is shown below.

$$\text{For } c > 0,\ a \le b \Rightarrow ac \le bc$$

$$\text{For } c < 0,\ a \le b \Rightarrow ac \ge bc$$

On your own, try to verify each of the properties of inequalities by using several examples with real numbers.

Solving a Linear Inequality in One Variable

The simplest type of inequality is a **linear inequality** in one variable. For instance,

$$2x + 3 > 4$$

is a linear inequality in x.

EXAMPLE 2 Solving a Linear Inequality

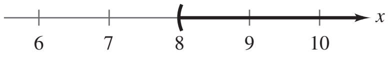
Solve $5x - 7 > 3x + 9$. Then graph the solution set.

REMARK Checking the solution set of an inequality is not as simple as checking the solutions of an equation. You can, however, get an indication of the validity of a solution set by substituting a few convenient values of x. For instance, in Example 2, try substituting $x = 5$ and $x = 10$ into the original inequality.

Solution

$$\begin{aligned} 5x - 7 &> 3x + 9 && \text{Write original inequality.} \\ 2x - 7 &> 9 && \text{Subtract } 3x \text{ from each side.} \\ 2x &> 16 && \text{Add 7 to each side.} \\ x &> 8 && \text{Divide each side by 2.} \end{aligned}$$

The solution set is all real numbers that are greater than 8, which is denoted by $(8, \infty)$. The graph of this solution set is shown below. Note that a parenthesis at 8 on the real number line indicates that 8 *is not* part of the solution set.



Solution interval: $(8, \infty)$

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Solve $7x - 3 \le 2x + 7$. Then graph the solution set.

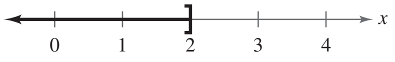
EXAMPLE 3 Solving a Linear Inequality

Solve $1 - \frac{3}{2}x \ge x - 4$.

Algebraic Solution

$$\begin{aligned} 1 - \frac{3x}{2} &\ge x - 4 && \text{Write original inequality.} \\ 2 - 3x &\ge 2x - 8 && \text{Multiply each side by 2.} \\ 2 - 5x &\ge -8 && \text{Subtract } 2x \text{ from each side.} \\ -5x &\ge -10 && \text{Subtract 2 from each side.} \\ x &\le 2 && \text{Divide each side by } -5 \text{ and reverse the inequality symbol.} \end{aligned}$$

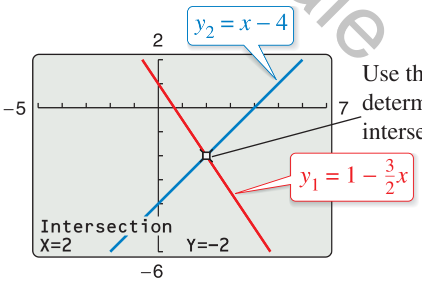
The solution set is all real numbers that are less than or equal to 2, which is denoted by $(-\infty, 2]$. The graph of this solution set is shown below. Note that a bracket at 2 on the real number line indicates that 2 *is* part of the solution set.



Solution interval: $(-\infty, 2]$

Graphical Solution

Use a graphing utility to graph $y_1 = 1 - \frac{3}{2}x$ and $y_2 = x - 4$ in the same viewing window, as shown below.



Use the *intersect* feature to determine that the graphs intersect at (2, −2).

The graph of y_1 lies above the graph of y_2 to the left of their point of intersection, which implies that $y_1 \ge y_2$ for all $x \le 2$.

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Solve $2 - \frac{5}{3}x > x - 6$ (a) algebraically and (b) graphically.

Sometimes it is possible to write two inequalities as a **double inequality.** For instance, you can write the two inequalities

$$-4 \le 5x - 2$$

and

$$5x - 2 < 7$$

more simply as

$$-4 \le 5x - 2 < 7. \qquad \text{Double inequality}$$

This form allows you to solve the two inequalities together, as demonstrated in Example 4.

EXAMPLE 4 Solving a Double Inequality

Solve $-3 \le 6x - 1 < 3$. Then graph the solution set.

Solution To solve a double inequality, you can isolate x as the middle term.

$$-3 \le 6x - 1 < 3 \qquad \text{Write original inequality.}$$

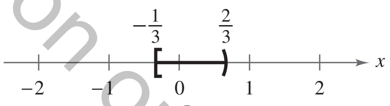
$$-3 + 1 \le 6x - 1 + 1 < 3 + 1 \qquad \text{Add 1 to each part.}$$

$$-2 \le 6x < 4 \qquad \text{Simplify.}$$

$$\frac{-2}{6} \le \frac{6x}{6} < \frac{4}{6} \qquad \text{Divide each part by 6.}$$

$$-\frac{1}{3} \le x < \frac{2}{3} \qquad \text{Simplify.}$$

The solution set is all real numbers that are greater than or equal to $-\frac{1}{3}$ and less than $\frac{2}{3}$, which is denoted by $\left[-\frac{1}{3}, \frac{2}{3}\right)$. The graph of this solution set is shown below.



Solution interval: $\left[-\frac{1}{3}, \frac{2}{3}\right)$

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Solve $1 < 2x + 7 < 11$. Then graph the solution set.

You can solve the double inequality in Example 4 in two parts, as follows.

$$-3 \le 6x - 1 \quad \text{and} \quad 6x - 1 < 3$$

$$-2 \le 6x \qquad\qquad 6x < 4$$

$$-\frac{1}{3} \le x \qquad\qquad x < \frac{2}{3}$$

The solution set consists of all real numbers that satisfy *both* inequalities. In other words, the solution set is the set of all values of x for which

$$-\frac{1}{3} \le x < \frac{2}{3}.$$

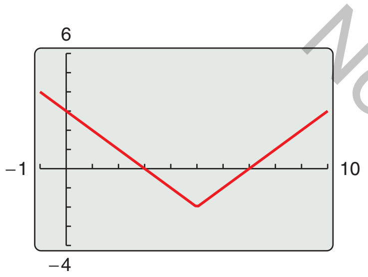
When combining two inequalities to form a double inequality, be sure that the inequalities satisfy the Transitive Property. For instance, it is *incorrect* to combine the inequalities $3 < x$ and $x \le -1$ as $3 < x \le -1$. This “inequality” is wrong because 3 is not less than -1.

Absolute Value Inequalities

TECHNOLOGY A graphing utility can be used to identify the solution set of an inequality. For instance, to find the solution set of $|x - 5| < 2$ (see Example 5a), rewrite the inequality as $|x - 5| - 2 < 0$, enter

Y1 = abs (X − 5) − 2

and press the *graph* key. The graph should look like the one shown below.



Notice that the graph is below the x-axis on the interval $(3, 7)$.

Solving an Absolute Value Inequality

Let u be an algebraic expression and let a be a real number such that $a > 0$.

1. $|u| < a$ if and only if $-a < u < a$.
2. $|u| \le a$ if and only if $-a \le u \le a$.
3. $|u| > a$ if and only if $u < -a$ or $u > a$.
4. $|u| \ge a$ if and only if $u \le -a$ or $u \ge a$.

EXAMPLE 5 Solving an Absolute Value Inequality

Solve each inequality. Then graph the solution set.

a. $|x - 5| < 2$ **b.** $|x + 3| \ge 7$

Solution

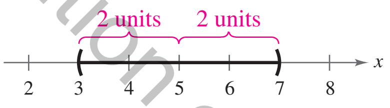
a. $|x - 5| < 2$ Write original inequality.

$-2 < x - 5 < 2$ Write equivalent inequalities.

$-2 + 5 < x - 5 + 5 < 2 + 5$ Add 5 to each part.

$3 < x < 7$ Simplify.

The solution set is all real numbers that are greater than 3 and less than 7, which is denoted by $(3, 7)$. The graph of this solution set is shown below. Note that the graph of the inequality can be described as all real numbers less than two units from 5.



$|x - 5| < 2$: Solutions lie inside $(3, 7)$.

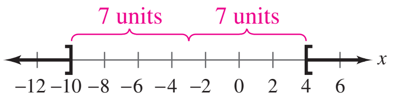
b. $|x + 3| \ge 7$ Write original inequality.

$x + 3 \le -7$ or $x + 3 \ge 7$ Write equivalent inequalities.

$x + 3 - 3 \le -7 - 3$ $x + 3 - 3 \ge 7 - 3$ Subtract 3 from each side.

$x \le -10$ $x \ge 4$ Simplify.

The solution set is all real numbers that are less than or equal to -10 *or* greater than or equal to 4, which is denoted by $(-\infty, -10] \cup [4, \infty)$. The symbol $\cup$ is the *union* symbol, which denotes the combining of two sets. The graph of this solution set is shown below. Note that the graph of the inequality can be described as all real numbers at least seven units from -3.



$|x + 3| \ge 7$: Solutions lie outside $(-10, 4)$.

Checkpoint *Audio-video solution in English & Spanish at LarsonPrecalculus.com.*

Solve $|x - 20| \le 4$. Then graph the solution set.

Applications

EXAMPLE 6 Comparative Shopping

Cell Phone Plans
Plan A:
$49.99 per month for 500 minutes plus $0.40 for each additional minute
Plan B:
$45.99 per month for 500 minutes plus $0.45 for each additional minute

Figure A.11

Consider the two cell phone plans shown in Figure A.11. How many *additional* minutes must you use in one month for plan B to cost more than plan A?

Solution Let m represent your additional minutes in one month. Write and solve an inequality.

$$\begin{aligned} 0.45m + 45.99 &> 0.40m + 49.99 \\ 0.05m &> 4 \\ m &> 80 \end{aligned}$$

Plan B costs more when you use more than 80 additional minutes in one month.

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Rework Example 6 when plan A costs $54.99 per month for 500 minutes plus $0.35 for each additional minute.

EXAMPLE 7 Accuracy of a Measurement

You buy chocolates that cost $9.89 per pound. The scale used to weigh your bag is accurate to within $\frac{1}{32}$ pound. According to the scale, your bag weighs $\frac{1}{2}$ pound and costs $4.95. How much might you have been undercharged or overcharged?

Solution Let x represent the actual weight of your bag. The difference of the actual weight and the weight shown on the scale is at most $\frac{1}{32}$ pound. That is, $\left|x - \frac{1}{2}\right| \le \frac{1}{32}$. You can solve this inequality as follows.

$$\begin{aligned} -\frac{1}{32} \le x - \frac{1}{2} &\le \frac{1}{32} \\ \frac{15}{32} \le x &\le \frac{17}{32} \end{aligned}$$

The least your bag can weigh is $\frac{15}{32}$ pound, which would have cost $4.64. The most the bag can weigh is $\frac{17}{32}$ pound, which would have cost $5.25. So, you might have been overcharged by as much as $0.31 or undercharged by as much as $0.30.

✓ ***Checkpoint*** *Audio-video solution in English & Spanish at LarsonPrecalculus.com.*

Rework Example 7 when the scale is accurate to within $\frac{1}{64}$ pound.

Summarize (Appendix A.6)

1. Describe how to use inequalities to represent intervals *(page A58, Example 1).*
2. State the properties of inequalities *(page A59).*
3. Describe how to solve a linear inequality *(pages A60 and A61, Examples 2–4).*
4. Describe how to solve an absolute value inequality *(page A62, Example 5).*
5. Describe a real-life example that uses an inequality *(page A63, Examples 6 and 7).*

A.6 Exercises

See **CalcChat.com** for tutorial help and worked-out solutions to odd-numbered exercises.

Vocabulary: Fill in the blanks.

1. The set of all real numbers that are solutions of an inequality is the ________ ________ of the inequality.

2. The set of all points on the real number line that represents the solution set of an inequality is the ________ of the inequality.

3. It is sometimes possible to write two inequalities as one inequality, called a ________ inequality.

4. The symbol ∪ is the ________ symbol, which denotes the combining of two sets.

Skills and Applications

Intervals and Inequalities In Exercises 5–12, write an inequality that represents the interval. Then state whether the interval is bounded or unbounded.

5. $[0, 9)$

6. $(-7, 4)$

7. $[-1, 5]$

8. $(2, 10]$

9. $(11, \infty)$

10. $[-5, \infty)$

11. $(-\infty, -2)$

12. $(-\infty, 7]$

Solving a Linear Inequality In Exercises 13–42, solve the inequality. Then graph the solution set.

13. $4x < 12$

14. $10x < -40$

15. $-2x > -3$

16. $-6x > 15$

17. $x - 5 \geq 7$

18. $x + 7 \leq 12$

19. $2x + 7 < 3 + 4x$

20. $3x + 1 \geq 2 + x$

21. $2x - 1 \geq 1 - 5x$

22. $6x - 4 \leq 2 + 8x$

23. $4 - 2x < 3(3 - x)$

24. $4(x + 1) < 2x + 3$

25. $\frac{3}{4}x - 6 \leq x - 7$

26. $3 + \frac{2}{7}x > x - 2$

27. $\frac{1}{2}(8x + 1) \geq 3x + \frac{5}{2}$

28. $9x - 1 < \frac{3}{4}(16x - 2)$

29. $3.6x + 11 \geq -3.4$

30. $15.6 - 1.3x < -5.2$

31. $1 < 2x + 3 < 9$

32. $-9 \leq -2x - 7 < 5$

33. $0 < 3(x + 7) \leq 20$

34. $-1 \leq -(x - 4) < 7$

35. $-4 < \dfrac{2x - 3}{3} < 4$

36. $0 \leq \dfrac{x + 3}{2} < 5$

37. $-1 < \dfrac{-x - 2}{3} \leq 1$

38. $-1 \leq \dfrac{-3x + 5}{7} \leq 2$

39. $\dfrac{3}{4} > x + 1 > \dfrac{1}{4}$

40. $-1 < 2 - \dfrac{x}{3} < 1$

41. $3.2 \leq 0.4x - 1 \leq 4.4$

42. $1.6 < 0.3x + 1 < 2.8$

Solving an Absolute Value Inequality In Exercises 43–58, solve the inequality. Then graph the solution set. (Some inequalities have no solution.)

43. $|x| < 5$

44. $|x| \geq 8$

45. $\left|\dfrac{x}{2}\right| > 1$

46. $\left|\dfrac{x}{5}\right| > 3$

47. $|x - 5| < -1$

48. $|x - 7| < -5$

49. $|x - 20| \leq 6$

50. $|x - 8| \geq 0$

51. $|3 - 4x| \geq 9$

52. $|1 - 2x| < 5$

53. $\left|\dfrac{x - 3}{2}\right| \geq 4$

54. $\left|1 - \dfrac{2x}{3}\right| < 1$

55. $|9 - 2x| - 2 < -1$

56. $|x + 14| + 3 > 17$

57. $2|x + 10| \geq 9$

58. $3|4 - 5x| \leq 9$

Graphical Analysis In Exercises 59–68, use a graphing utility to graph the inequality and identify the solution set.

59. $6x > 12$

60. $3x - 1 \leq 5$

61. $5 - 2x \geq 1$

62. $20 < 6x - 1$

63. $4(x - 3) \leq 8 - x$

64. $3(x + 1) < x + 7$

65. $|x - 8| \leq 14$

66. $|2x + 9| > 13$

67. $2|x + 7| \geq 13$

68. $\frac{1}{2}|x + 1| \leq 3$

Graphical Analysis In Exercises 69–74, use a graphing utility to graph the equation. Use the graph to approximate the values of x that satisfy each inequality.

	Equation	Inequalities	
69.	$y = 2x - 3$	(a) $y \geq 1$	(b) $y \leq 0$
70.	$y = \frac{2}{3}x + 1$	(a) $y \leq 5$	(b) $y \geq 0$
71.	$y = -\frac{1}{2}x + 2$	(a) $0 \leq y \leq 3$	(b) $y \geq 0$
72.	$y = -3x + 8$	(a) $-1 \leq y \leq 3$	(b) $y \leq 0$
73.	$y = \lvert x - 3 \rvert$	(a) $y \leq 2$	(b) $y \geq 4$
74.	$y = \left\lvert \frac{1}{2}x + 1 \right\rvert$	(a) $y \leq 4$	(b) $y \geq 1$

Finding an Interval In Exercises 75–80, find the interval on the real number line for which the radicand is nonnegative.

75. $\sqrt{x - 5}$

76. $\sqrt{x - 10}$

77. $\sqrt{x + 3}$

78. $\sqrt{3 - x}$

79. $\sqrt[4]{7 - 2x}$

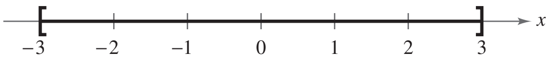
80. $\sqrt[4]{6x + 15}$

81. Think About It The graph of $|x - 5| < 3$ can be described as all real numbers less than three units from 5. Give a similar description of $|x - 10| < 8$.

82. **Think About It** The graph of $|x - 2| > 5$ can be described as all real numbers more than five units from 2. Give a similar description of $|x - 8| > 4$.

Using Absolute Value **In Exercises 83–90, use absolute value notation to define the interval (or pair of intervals) on the real number line.**

83.



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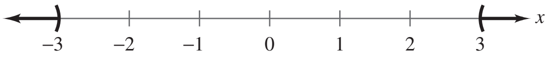

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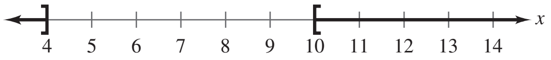
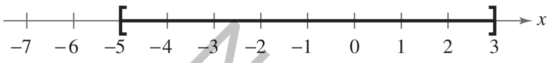


86.



87. All real numbers at least 10 units from 12

88. All real numbers at least five units from 8

89. All real numbers more than four units from -3

90. All real numbers no more than seven units from -6

Writing an Inequality **In Exercises 91–94, write an inequality to describe the situation.**

91. A company expects its earnings per share E for the next quarter to be no less than \$4.10 and no more than \$4.25.

92. The estimated daily oil production p at a refinery is greater than 2 million barrels but less than 2.4 million barrels.

93. The return r on an investment is expected to be no more than 8%.

94. The net income I of a company is expected to be no less than \$239 million.

Physiology **The maximum heart rate r (in beats per minute) of a person in normal health is related to the person's age A (in years) by the equation**

$$r = 220 - A.$$

In Exercises 95 and 96, determine the interval in which the person's heart rate is from 50% to 85% of the maximum heart rate. *(Source: American Heart Association)*

95. a 20-year-old

96. a 40-year-old

97. **Job Offers** You are considering two job offers. The first job pays \$13.50 per hour. The second job pays \$9.00 per hour plus \$0.75 per unit produced per hour. How many units must you produce per hour for the second job to pay more per hour than the first job?

98. **Job Offers** You are considering two job offers. The first job pays \$3000 per month. The second job pays \$1000 per month plus a commission of 4% of your gross sales. How much must you earn in gross sales for the second job to pay more per month than the first job?

99. **Investment** For what annual interest rates will an investment of \$1000 grow to more than \$1062.50 in 2 years? $[A = P(1 + rt)]$

100. **Investment** For what annual interest rates will an investment of \$750 grow to more than \$825 in 2 years? $[A = P(1 + rt)]$

101. **Cost, Revenue, and Profit** The revenue from selling x units of a product is $R = 115.95x$. The cost of producing x units is $C = 95x + 750$. To obtain a profit, the revenue must be greater than the cost. For what values of x will this product return a profit?

102. **Cost, Revenue, and Profit** The revenue from selling x units of a product is $R = 24.55x$. The cost of producing x units is $C = 15.4x + 150{,}000$. To obtain a profit, the revenue must be greater than the cost. For what values of x will this product return a profit?

103. **Daily Sales** A doughnut shop sells a dozen doughnuts for \$7.95. Beyond the fixed costs (rent, utilities, and insurance) of \$165 per day, it costs \$1.45 for enough materials (flour, sugar, and so on) and labor to produce a dozen doughnuts. The daily profit from doughnut sales varies between \$400 and \$1200. Between what levels (in dozens of doughnuts) do the daily sales vary?

104. **Weight Loss Program** A person enrolls in a diet and exercise program that guarantees a loss of at least $1\frac{1}{2}$ pounds per week. The person's weight at the beginning of the program is 164 pounds. Find the maximum number of weeks before the person attains a goal weight of 128 pounds.

105. **Data Analysis: IQ Scores and GPA** The admissions office of a college wants to determine whether there is a relationship between IQ scores x and grade-point averages y after the first year of school. An equation that models the data the admissions office obtained is $y = 0.067x - 5.638$.

(a) Use a graphing utility to graph the model.

(b) Use the graph to estimate the values of x that predict a grade-point average of at least 3.0.

(c) Verify your estimate from part (b) algebraically.

(d) List other factors that may influence GPA.

106. **Data Analysis: Weightlifting** You want to determine whether there is a relationship between an athlete's weight x (in pounds) and the athlete's maximum bench-press weight y (in pounds). An equation that models the data you obtained is $y = 1.3x - 36$.

(a) Use a graphing utility to graph the model.

(b) Use the graph to estimate the values of x that predict a maximum bench-press weight of at least 200 pounds.

(c) Verify your estimate from part (b) algebraically.

(d) List other factors that might influence an individual's maximum bench-press weight.

107. Teachers' Salaries The average salaries S (in thousands of dollars) for public elementary school teachers in the United States from 2001 through 2011 can be modeled by

$$S = 1.36t + 41.1, \quad 1 \le t \le 11$$

where t represents the year, with $t = 1$ corresponding to 2001. *(Source: National Education Association)*

(a) According to the model, when was the average salary at least \$45,000, but no more than \$50,000?

(b) Use the model to predict when the average salary will exceed \$62,000.

108. Milk Production Milk production M (in billions of pounds) in the United States from 2002 through 2010 can be modeled by

$$M = 3.24t + 161.5, \quad 2 \le t \le 10$$

where t represents the year, with $t = 2$ corresponding to 2002. *(Source: U.S. Department of Agriculture)*

(a) According to the model, when was the annual milk production greater than 178 billion pounds, but no more than 190 billion pounds?

(b) Use the model to predict when milk production will exceed 225 billion pounds.

109. Geometry The side of a square is measured as 10.4 inches with a possible error of $\frac{1}{16}$ inch. Using these measurements, determine the interval containing the possible areas of the square.

110. Geometry The side of a square is measured as 24.2 centimeters with a possible error of 0.25 centimeter. Using these measurements, determine the interval containing the possible areas of the square.

111. Accuracy of Measurement You stop at a self-service gas station to buy 15 gallons of 87-octane gasoline at \$3.61 per gallon. The gas pump is accurate to within $\frac{1}{10}$ gallon. How much might you be undercharged or overcharged?

112. Accuracy of Measurement You buy six T-bone steaks that cost \$8.99 per pound. The weight that is listed on the package is 5.72 pounds. The scale that weighed the package is accurate to within $\frac{1}{32}$ pound. How much might you be undercharged or overcharged?

113. Time Study A time study was conducted to determine the length of time required to perform a particular task in a manufacturing process. The times required by approximately two-thirds of the workers in the study satisfied the inequality

$$|t - 15.6| \le 1.9$$

where t is time in minutes. Determine the interval in which these times lie.

Jaroslav74/Shutterstock.com

114. Error Tolerance

The protective cover layer of a Blu-ray Disc™ is 100 micrometers thick with an error tolerance of 3 micrometers. Write an absolute value inequality for the possible thicknesses of the cover layer. Then graph the solution set. *(Source: Blu-ray Disc™ Association)*

Exploration

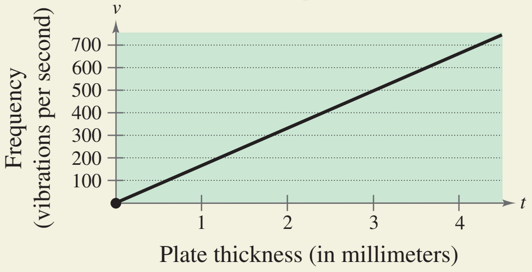
True or False? **In Exercises 115–117, determine whether the statement is true or false. Justify your answer.**

115. If a, b, and c are real numbers, and $a < b$, then $a + c < b + c$.

116. If a, b, and c are real numbers, and $a \le b$, then $ac \le bc$.

117. If $-10 \le x \le 8$, then $-10 \ge -x$ and $-x \ge -8$.

118. HOW DO YOU SEE IT? Michael Kasha of Florida State University used physics and mathematics to design a new classical guitar. The model he used for the frequency of the vibrations on a circular plate was $v = (2.6t/d^2)\sqrt{E/\rho}$, where v is the frequency (in vibrations per second), t is the plate thickness (in millimeters), d is the diameter of the plate, E is the elasticity of the plate material, and ρ is the density of the plate material. For fixed values of d, E, and ρ, the graph of the equation is a line (see figure).



(a) Estimate the frequency when the plate thickness is 2 millimeters.

(b) Approximate the interval for the frequency when the plate thickness is greater than or equal to 0 millimeters and less than 3 millimeters.

A.7 Errors and the Algebra of Calculus

- **Avoid common algebraic errors.**
- **Recognize and use algebraic techniques that are common in calculus.**

Algebraic Errors to Avoid

This section contains five lists of common algebraic errors: errors involving parentheses, errors involving fractions, errors involving exponents, errors involving radicals, and errors involving dividing out. Many of these errors are made because they seem to be the *easiest* things to do. For instance, the operations of subtraction and division are often believed to be commutative and associative. The following examples illustrate the fact that subtraction and division are neither commutative nor associative.

Not commutative	**Not associative**
$4 - 3 \neq 3 - 4$	$8 - (6 - 2) \neq (8 - 6) - 2$
$15 \div 5 \neq 5 \div 15$	$20 \div (4 \div 2) \neq (20 \div 4) \div 2$

Errors Involving Parentheses

Potential Error	**Correct Form**	**Comment**
~~$a - (x - b) = a - x - b$~~	$a - (x - b) = a - x + b$	Distribute to each term in parentheses.
~~$(a + b)^2 = a^2 + b^2$~~	$(a + b)^2 = a^2 + 2ab + b^2$	Remember the middle term when squaring binomials.
$\xcancel{\left(\frac{1}{2}a\right)\left(\frac{1}{2}b\right) = \frac{1}{2}(ab)}$	$\left(\frac{1}{2}a\right)\left(\frac{1}{2}b\right) = \frac{1}{4}(ab) = \frac{ab}{4}$	$\frac{1}{2}$ occurs twice as a factor.
~~$(3x + 6)^2 = 3(x + 2)^2$~~	$(3x + 6)^2 = [3(x + 2)]^2$ $= 3^2(x + 2)^2$	When factoring, raise all factors to the power.

Errors Involving Fractions

Potential Error	**Correct Form**	**Comment**
$\xcancel{\frac{2}{x + 4} = \frac{2}{x} + \frac{2}{4}}$	Leave as $\frac{2}{x + 4}$.	The fraction is already in simplest form.
$\xcancel{\frac{\left(\frac{x}{a}\right)}{b} = \frac{bx}{a}}$	$\frac{\left(\frac{x}{a}\right)}{b} = \left(\frac{x}{a}\right)\left(\frac{1}{b}\right) = \frac{x}{ab}$	Multiply by the reciprocal when dividing fractions.
$\xcancel{\frac{1}{a} + \frac{1}{b} = \frac{1}{a + b}}$	$\frac{1}{a} + \frac{1}{b} = \frac{b + a}{ab}$	Use the property for adding fractions.
$\xcancel{\frac{1}{3x} = \frac{1}{3}x}$	$\frac{1}{3x} = \frac{1}{3} \cdot \frac{1}{x}$	Use the property for multiplying fractions.
$\xcancel{(1/3)x = \frac{1}{3x}}$	$(1/3)x = \frac{1}{3} \cdot x = \frac{x}{3}$	Be careful when expressing fractions in the form $1/a$.
$\xcancel{(1/x) + 2 = \frac{1}{x + 2}}$	$(1/x) + 2 = \frac{1}{x} + 2 = \frac{1 + 2x}{x}$	Be careful when expressing fractions in the form $1/a$. Be sure to find a common denominator before adding fractions.

Errors Involving Exponents

Potential Error	Correct Form	Comment
~~$(x^2)^3 = x^5$~~	$(x^2)^3 = x^{2\cdot 3} = x^6$	Multiply exponents when raising a power to a power.
~~$x^2 \cdot x^3 = x^6$~~	$x^2 \cdot x^3 = x^{2+3} = x^5$	Add exponents when multiplying powers with like bases.
~~$(2x)^3 = 2x^3$~~	$(2x)^3 = 2^3x^3 = 8x^3$	Raise each factor to the power.
~~$\frac{1}{x^2 - x^3} = x^{-2} - x^{-3}$~~	Leave as $\frac{1}{x^2 - x^3}$.	Do not move term-by-term from denominator to numerator.

Errors Involving Radicals

Potential Error	Correct Form	Comment
~~$\sqrt{5x} = 5\sqrt{x}$~~	$\sqrt{5x} = \sqrt{5}\sqrt{x}$	Radicals apply to every factor inside the radical.
~~$\sqrt{x^2 + a^2} = x + a$~~	Leave as $\sqrt{x^2 + a^2}$.	Do not apply radicals term-by-term when adding or subtracting terms.
~~$\sqrt{-x + a} = -\sqrt{x - a}$~~	Leave as $\sqrt{-x + a}$.	Do not factor minus signs out of square roots.

Errors Involving Dividing Out

Potential Error	Correct Form	Comment
~~$\frac{a + bx}{a} = 1 + bx$~~	$\frac{a + bx}{a} = \frac{a}{a} + \frac{bx}{a} = 1 + \frac{b}{a}x$	Divide out common factors, not common terms.
~~$\frac{a + ax}{a} = a + x$~~	$\frac{a + ax}{a} = \frac{a(1 + x)}{a} = 1 + x$	Factor before dividing out common factors.
~~$1 + \frac{x}{2x} = 1 + \frac{1}{x}$~~	$1 + \frac{x}{2x} = 1 + \frac{1}{2} = \frac{3}{2}$	Divide out common factors.

A good way to avoid errors is to *work slowly, write neatly,* and *talk to yourself.* Each time you write a step, ask yourself why the step is algebraically legitimate. You can justify the step below because *dividing the numerator and denominator by the same nonzero number produces an equivalent fraction.*

$$\frac{2x}{6} = \frac{\cancel{2} \cdot x}{\cancel{2} \cdot 3} = \frac{x}{3}$$

EXAMPLE 1 Describing and Correcting an Error

Describe and correct the error. ~~$\frac{1}{2x} + \frac{1}{3x} = \frac{1}{5x}$~~

Solution Use the property for adding fractions: $\frac{1}{a} + \frac{1}{b} = \frac{b + a}{ab}$.

$$\frac{1}{2x} + \frac{1}{3x} = \frac{3x + 2x}{6x^2} = \frac{5x}{6x^2} = \frac{5}{6x}$$

✓ ***Checkpoint*** *Audio-video solution in English & Spanish at LarsonPrecalculus.com.*

Describe and correct the error. ~~$\sqrt{x^2 + 4} = x + 2$~~

Some Algebra of Calculus

In calculus it is often necessary to take a simplified algebraic expression and rewrite it. See the following lists, taken from a standard calculus text.

Unusual Factoring

Expression	**Useful Calculus Form**	**Comment**
$\frac{5x^4}{8}$	$\frac{5}{8}x^4$	Write with fractional coefficient.
$\frac{x^2 + 3x}{-6}$	$-\frac{1}{6}(x^2 + 3x)$	Write with fractional coefficient.
$2x^2 - x - 3$	$2\left(x^2 - \frac{x}{2} - \frac{3}{2}\right)$	Factor out the leading coefficient.
$\frac{x}{2}(x + 1)^{-1/2} + (x + 1)^{1/2}$	$\frac{(x + 1)^{-1/2}}{2}[x + 2(x + 1)]$	Factor out the variable expression with the lesser exponent.

Writing with Negative Exponents

Expression	**Useful Calculus Form**	**Comment**
$\frac{9}{5x^3}$	$\frac{9}{5}x^{-3}$	Move the factor to the numerator and change the sign of the exponent.
$\frac{7}{\sqrt{2x - 3}}$	$7(2x - 3)^{-1/2}$	Move the factor to the numerator and change the sign of the exponent.

Writing a Fraction as a Sum

Expression	**Useful Calculus Form**	**Comment**
$\frac{x + 2x^2 + 1}{\sqrt{x}}$	$x^{1/2} + 2x^{3/2} + x^{-1/2}$	Divide each term of the numerator by $x^{1/2}$.
$\frac{1 + x}{x^2 + 1}$	$\frac{1}{x^2 + 1} + \frac{x}{x^2 + 1}$	Rewrite the fraction as a sum of fractions.
$\frac{2x}{x^2 + 2x + 1}$	$\frac{2x + 2 - 2}{x^2 + 2x + 1}$	Add and subtract the same term.
	$= \frac{2x + 2}{x^2 + 2x + 1} - \frac{2}{(x + 1)^2}$	Rewrite the fraction as a difference of fractions.
$\frac{x^2 - 2}{x + 1}$	$x - 1 - \frac{1}{x + 1}$	Use long division. (See Section 2.3.)
$\frac{x + 7}{x^2 - x - 6}$	$\frac{2}{x - 3} - \frac{1}{x + 2}$	Use the method of partial fractions. (See Section 7.4.)

Inserting Factors and Terms

Expression	**Useful Calculus Form**	**Comment**
$(2x - 1)^3$	$\frac{1}{2}(2x - 1)^3(2)$	Multiply and divide by 2.
$7x^2(4x^3 - 5)^{1/2}$	$\frac{7}{12}(4x^3 - 5)^{1/2}(12x^2)$	Multiply and divide by 12.
$\frac{4x^2}{9} - 4y^2 = 1$	$\frac{x^2}{9/4} - \frac{y^2}{1/4} = 1$	Write with fractional denominators.
$\frac{x}{x + 1}$	$\frac{x + 1 - 1}{x + 1} = 1 - \frac{1}{x + 1}$	Add and subtract the same term.

The next five examples demonstrate many of the steps in the preceding lists.

EXAMPLE 2 Factors Involving Negative Exponents

Factor $x(x + 1)^{-1/2} + (x + 1)^{1/2}$.

Solution When multiplying powers with like bases, you add exponents. When factoring, you are undoing multiplication, and so you *subtract* exponents.

$$\begin{aligned} x(x + 1)^{-1/2} + (x + 1)^{1/2} &= (x + 1)^{-1/2}[x(x + 1)^0 + (x + 1)^1] \\ &= (x + 1)^{-1/2}[x + (x + 1)] \\ &= (x + 1)^{-1/2}(2x + 1) \end{aligned}$$

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Factor $x(x - 2)^{-1/2} + 3(x - 2)^{1/2}$.

Another way to simplify the expression in Example 2 is to multiply the expression by a fractional form of 1 and then use the Distributive Property.

$$\begin{aligned} x(x + 1)^{-1/2} + (x + 1)^{1/2} &= [x(x + 1)^{-1/2} + (x + 1)^{1/2}] \cdot \frac{(x + 1)^{1/2}}{(x + 1)^{1/2}} \\ &= \frac{x(x + 1)^0 + (x + 1)^1}{(x + 1)^{1/2}} = \frac{2x + 1}{\sqrt{x + 1}} \end{aligned}$$

EXAMPLE 3 Inserting Factors in an Expression

Insert the required factor: $\frac{x + 2}{(x^2 + 4x - 3)^2} = (\quad)\frac{1}{(x^2 + 4x - 3)^2}(2x + 4)$.

Solution The expression on the right side of the equation is twice the expression on the left side. To make both sides equal, insert a factor of $\frac{1}{2}$.

$$\frac{x + 2}{(x^2 + 4x - 3)^2} = \left(\frac{1}{2}\right)\frac{1}{(x^2 + 4x - 3)^2}(2x + 4)$$

Right side is multiplied and divided by 2.

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Insert the required factor: $\frac{6x - 3}{(x^2 - x + 4)^2} = (\quad)\frac{1}{(x^2 - x + 4)^2}(2x - 1)$.

EXAMPLE 4 Rewriting Fractions

Explain why the two expressions are equivalent.

$$\frac{4x^2}{9} - 4y^2 = \frac{x^2}{9/4} - \frac{y^2}{1/4}$$

Solution To write the expression on the left side of the equation in the form given on the right side, multiply the numerator and denominator of each term by $\frac{1}{4}$.

$$\frac{4x^2}{9} - 4y^2 = \frac{4x^2}{9}\left(\frac{1/4}{1/4}\right) - 4y^2\left(\frac{1/4}{1/4}\right) = \frac{x^2}{9/4} - \frac{y^2}{1/4}$$

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Explain why the two expressions are equivalent.

$$\frac{9x^2}{16} + 25y^2 = \frac{x^2}{16/9} + \frac{y^2}{1/25}$$

EXAMPLE 5 Rewriting with Negative Exponents

Rewrite each expression using negative exponents.

a. $\dfrac{-4x}{(1 - 2x^2)^2}$ **b.** $\dfrac{2}{5x^3} - \dfrac{1}{\sqrt{x}} + \dfrac{3}{5(4x)^2}$

Solution

a. $\dfrac{-4x}{(1 - 2x^2)^2} = -4x(1 - 2x^2)^{-2}$

b.
$$\frac{2}{5x^3} - \frac{1}{\sqrt{x}} + \frac{3}{5(4x)^2} = \frac{2}{5x^3} - \frac{1}{x^{1/2}} + \frac{3}{5(4x)^2}$$
$$= \frac{2}{5}x^{-3} - x^{-1/2} + \frac{3}{5}(4x)^{-2}$$

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Rewrite $\dfrac{-6x}{(1 - 3x^2)^2} + \dfrac{1}{\sqrt[3]{x}}$ using negative exponents.

EXAMPLE 6 Rewriting a Fraction as a Sum of Terms

Rewrite each fraction as the sum of three terms.

a. $\dfrac{x^2 - 4x + 8}{2x}$ **b.** $\dfrac{x + 2x^2 + 1}{\sqrt{x}}$

Solution

a.
$$\frac{x^2 - 4x + 8}{2x} = \frac{x^2}{2x} - \frac{4x}{2x} + \frac{8}{2x}$$
$$= \frac{x}{2} - 2 + \frac{4}{x}$$

b.
$$\frac{x + 2x^2 + 1}{\sqrt{x}} = \frac{x}{x^{1/2}} + \frac{2x^2}{x^{1/2}} + \frac{1}{x^{1/2}}$$
$$= x^{1/2} + 2x^{3/2} + x^{-1/2}$$

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Rewrite $\dfrac{x^4 - 2x^3 + 5}{x^3}$ as the sum of three terms.

A.7 Exercises

See **CalcChat.com** for tutorial help and worked-out solutions to odd-numbered exercises.

Vocabulary: Fill in the blanks.

1. To rewrite the expression $\frac{3}{x^5}$ using negative exponents, move x^5 to the ________ and change the sign of the exponent.

2. When dividing fractions, multiply by the ________.

Skills and Applications

Describing and Correcting an Error **In Exercises 3–22, describe and correct the error.**

3. $\xcancel{2x - (3y + 4) = 2x - 3y + 4}$

4. $\xcancel{5z + 3(x - 2) = 5z + 3x - 2}$

5. $\xcancel{\frac{4}{16x - (2x + 1)} = \frac{4}{14x + 1}}$

6. $\xcancel{\frac{1 - x}{(5 - x)(-x)} = \frac{x - 1}{x(x - 5)}}$

7. $\xcancel{(5z)(6z) = 30z}$

8. $\xcancel{x(yz) = (xy)(xz)}$

9. $\xcancel{a\left(\frac{x}{y}\right) = \frac{ax}{ay}}$

10. $\xcancel{(4x)^2 = 4x^2}$

11. $\xcancel{\sqrt{x + 9} = \sqrt{x} + 3}$

12. $\xcancel{\sqrt{25 - x^2} = 5 - x}$

13. $\xcancel{\frac{2x^2 + 1}{5x} = \frac{2x + 1}{5}}$

14. $\xcancel{\frac{6x + y}{6x - y} = \frac{x + y}{x - y}}$

15. $\xcancel{\frac{1}{a^{-1} + b^{-1}} = \left(\frac{1}{a + b}\right)^{-1}}$

16. $\xcancel{\frac{1}{x + y^{-1}} = \frac{y}{x + 1}}$

17. $\xcancel{(x^2 + 5x)^{1/2} = x(x + 5)^{1/2}}$

18. $\xcancel{x(2x - 1)^2 = (2x^2 - x)^2}$

19. $\xcancel{\frac{3}{x} + \frac{4}{y} = \frac{7}{x + y}}$

20. $\xcancel{\frac{1}{2y} = (1/2)y}$

21. $\xcancel{\frac{x}{2y} + \frac{y}{3} = \frac{x + y}{2y + 3}}$

22. $\xcancel{5 + (1/y) = \frac{1}{5 + y}}$

Inserting Factors in an Expression **In Exercises 23–44, insert the required factor in the parentheses.**

23. $\frac{5x + 3}{4} = \frac{1}{4}(\square)$

24. $\frac{7x^2}{10} = \frac{7}{10}(\square)$

25. $\frac{2}{3}x^2 + \frac{1}{3}x + 5 = \frac{1}{3}(\square)$

26. $\frac{3}{4}x + \frac{1}{2} = \frac{1}{4}(\square)$

27. $x^2(x^3 - 1)^4 = (\square)(x^3 - 1)^4(3x^2)$

28. $x(1 - 2x^2)^3 = (\square)(1 - 2x^2)^3(-4x)$

29. $2(y - 5)^{1/2} + y(y - 5)^{-1/2} = (y - 5)^{-1/2}(\square)$

30. $3t(6t + 1)^{-1/2} + (6t + 1)^{1/2} = (6t + 1)^{-1/2}(\square)$

31. $\frac{4x + 6}{(x^2 + 3x + 7)^3} = (\square)\frac{1}{(x^2 + 3x + 7)^3}(2x + 3)$

32. $\frac{x + 1}{(x^2 + 2x - 3)^2} = (\square)\frac{1}{(x^2 + 2x - 3)^2}(2x + 2)$

33. $\frac{3}{x} + \frac{5}{2x^2} - \frac{3}{2}x = (\square)(6x + 5 - 3x^3)$

34. $\frac{(x - 1)^2}{169} + (y + 5)^2 = \frac{(x - 1)^3}{169(\square)} + (y + 5)^2$

35. $\frac{25x^2}{36} + \frac{4y^2}{9} = \frac{x^2}{(\square)} + \frac{y^2}{(\square)}$

36. $\frac{5x^2}{9} - \frac{16y^2}{49} = \frac{x^2}{(\square)} - \frac{y^2}{(\square)}$

37. $\frac{x^2}{3/10} - \frac{y^2}{4/5} = \frac{10x^2}{(\square)} - \frac{5y^2}{(\square)}$

38. $\frac{x^2}{5/8} + \frac{y^2}{6/11} = \frac{8x^2}{(\square)} + \frac{11y^2}{(\square)}$

39. $x^{1/3} - 5x^{4/3} = x^{1/3}(\square)$

40. $3(2x + 1)x^{1/2} + 4x^{3/2} = x^{1/2}(\square)$

41. $(1 - 3x)^{4/3} - 4x(1 - 3x)^{1/3} = (1 - 3x)^{1/3}(\square)$

42. $\frac{1}{2\sqrt{x}} + 5x^{3/2} - 10x^{5/2} = \frac{1}{2\sqrt{x}}(\square)$

43. $\frac{1}{10}(2x + 1)^{5/2} - \frac{1}{6}(2x + 1)^{3/2} = \frac{(2x + 1)^{3/2}}{15}(\square)$

44. $\frac{3}{7}(t + 1)^{7/3} - \frac{3}{4}(t + 1)^{4/3} = \frac{3(t + 1)^{4/3}}{28}(\square)$

Rewriting with Negative Exponents **In Exercises 45–50, rewrite the expression using negative exponents.**

45. $\frac{7}{(x + 3)^5}$

46. $\frac{2 - x}{(x + 1)^{3/2}}$

47. $\frac{2x^5}{(3x + 5)^4}$

48. $\frac{x + 1}{x(6 - x)^{1/2}}$

49. $\frac{4}{3x} + \frac{4}{x^4} - \frac{7x}{\sqrt[3]{2x}}$

50. $\frac{x}{x - 2} + \frac{1}{x^2} + \frac{8}{3(9x)^3}$

Rewriting a Fraction as a Sum of Terms **In Exercises 51–56, rewrite the fraction as a sum of two or more terms.**

51. $\frac{x^2 + 6x + 12}{3x}$

52. $\frac{x^3 - 5x^2 + 4}{x^2}$

53. $\frac{4x^3 - 7x^2 + 1}{x^{1/3}}$

54. $\frac{2x^5 - 3x^3 + 5x - 1}{x^{3/2}}$

55. $\frac{3 - 5x^2 - x^4}{\sqrt{x}}$

56. $\frac{x^3 - 5x^4}{3x^2}$

Simplifying an Expression **In Exercises 57–68, simplify the expression.**

57. $\dfrac{-2(x^2 - 3)^{-3}(2x)(x + 1)^3 - 3(x + 1)^2(x^2 - 3)^{-2}}{[(x + 1)^3]^2}$

58. $\dfrac{x^5(-3)(x^2 + 1)^{-4}(2x) - (x^2 + 1)^{-3}(5)x^4}{(x^5)^2}$

59. $\dfrac{(6x + 1)^3(27x^2 + 2) - (9x^3 + 2x)(3)(6x + 1)^2(6)}{[(6x + 1)^3]^2}$

60. $\dfrac{(4x^2 + 9)^{1/2}(2) - (2x + 3)\left(\frac{1}{2}\right)(4x^2 + 9)^{-1/2}(8x)}{[(4x^2 + 9)^{1/2}]^2}$

61. $\dfrac{(x + 2)^{3/4}(x + 3)^{-2/3} - (x + 3)^{1/3}(x + 2)^{-1/4}}{[(x + 2)^{3/4}]^2}$

62. $(2x - 1)^{1/2} - (x + 2)(2x - 1)^{-1/2}$

63. $\dfrac{2(3x - 1)^{1/3} - (2x + 1)\left(\frac{1}{3}\right)(3x - 1)^{-2/3}(3)}{(3x - 1)^{2/3}}$

64. $\dfrac{(x + 1)\left(\frac{1}{2}\right)(2x - 3x^2)^{-1/2}(2 - 6x) - (2x - 3x^2)^{1/2}}{(x + 1)^2}$

65. $\dfrac{1}{(x^2 + 4)^{1/2}} \cdot \dfrac{1}{2}(x^2 + 4)^{-1/2}(2x)$

66. $\dfrac{1}{x^2 - 6}(2x) + \dfrac{1}{2x + 5}(2)$

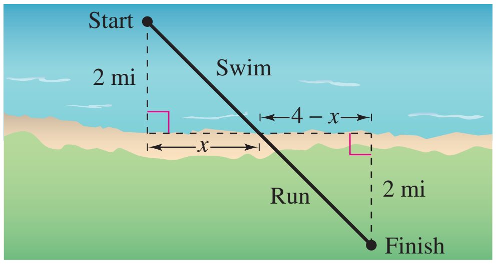
67. $(x^2 + 5)^{1/2}\left(\frac{3}{2}\right)(3x - 2)^{1/2}(3)$
$\qquad + (3x - 2)^{3/2}\left(\frac{1}{2}\right)(x^2 + 5)^{-1/2}(2x)$

68. $(3x + 2)^{-1/2}(3)(x - 6)^{1/2}(1)$
$\qquad + (x - 6)^3\left(-\frac{1}{2}\right)(3x + 2)^{-3/2}(3)$

69. Athletics An athlete has set up a course for training as part of her regimen in preparation for an upcoming triathlon. She is dropped off by a boat 2 miles from the nearest point on shore. The finish line is 4 miles down the coast and 2 miles inland (see figure). She can swim 2 miles per hour and run 6 miles per hour. The time t (in hours) required for her to reach the finish line can be approximated by the model

$$t = \frac{\sqrt{x^2 + 4}}{2} + \frac{\sqrt{(4 - x)^2 + 4}}{6}$$

where x is the distance down the coast (in miles) to the point at which she swims and then leaves the water to start her run.



(a) Find the times required for the triathlete to finish when she swims to the points $x = 0.5$, $x = 1.0, \ldots, x = 3.5$, and $x = 4.0$ miles down the coast.

(b) Use your results from part (a) to determine the distance down the coast that will yield the minimum amount of time required for the triathlete to reach the finish line.

(c) The expression below was obtained using calculus. It can be used to find the minimum amount of time required for the triathlete to reach the finish line. Simplify the expression.

$$\tfrac{1}{2}x(x^2 + 4)^{-1/2} + \tfrac{1}{6}(x - 4)(x^2 - 8x + 20)^{-1/2}$$

70. Verifying an Equation

(a) Verify that $y_1 = y_2$ analytically.

$$y_1 = x^2\left(\frac{1}{3}\right)(x^2 + 1)^{-2/3}(2x) + (x^2 + 1)^{1/3}(2x)$$

$$y_2 = \frac{2x(4x^2 + 3)}{3(x^2 + 1)^{2/3}}$$

(b) Complete the table and demonstrate the equality in part (a) numerically.

x	-2	-1	$-\frac{1}{2}$	0	1	2	$\frac{5}{2}$
y_1							
y_2							

Exploration

71. Writing Write a paragraph explaining to a classmate why $\dfrac{1}{(x - 2)^{1/2} + x^4} \neq (x - 2)^{-1/2} + x^{-4}$.

72. Think About It You are taking a course in calculus, and for one of the homework problems you obtain the following answer.

$$\frac{1}{10}(2x - 1)^{5/2} + \frac{1}{6}(2x - 1)^{3/2}$$

The answer in the back of the book is

$$\frac{1}{15}(2x - 1)^{3/2}(3x + 1).$$

Show how the second answer can be obtained from the first. Then use the same technique to simplify each of the following expressions.

(a) $\dfrac{2}{3}x(2x - 3)^{3/2} - \dfrac{2}{15}(2x - 3)^{5/2}$

(b) $\dfrac{2}{3}x(4 + x)^{3/2} - \dfrac{2}{15}(4 + x)^{5/2}$