

Appendix C: Review of Graphs, Equations, and Inequalities

C.1 The Cartesian Plane

The Cartesian Plane

Just as you can represent real numbers by points on a real number line, you can represent ordered pairs of real numbers by points in a plane called the **rectangular coordinate system**, or the **Cartesian plane**, after the French mathematician René Descartes (1596–1650).

The Cartesian plane is formed by using two real number lines intersecting at right angles, as shown in Figure C.1. The horizontal real number line is usually called the **x-axis**, and the vertical real number line is usually called the **y-axis**. The point of intersection of these two axes is the **origin**, and the two axes divide the plane into four parts called **quadrants**.

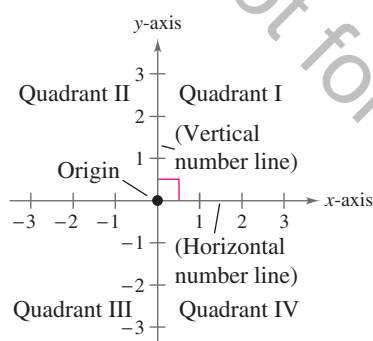


Figure C.1 The Cartesian Plane

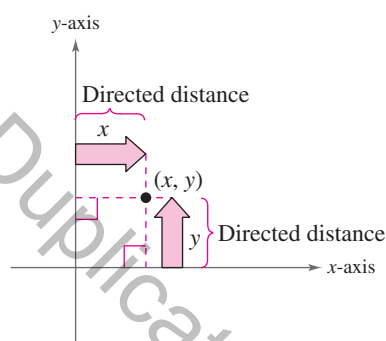


Figure C.2 Ordered Pair (x, y)

Each point in the plane corresponds to an **ordered pair** (x, y) of real numbers x and y , called **coordinates** of the point. The **x-coordinate** represents the directed distance from the y-axis to the point, and the **y-coordinate** represents the directed distance from the x-axis to the point, as shown in Figure C.2.

Directed distance from y-axis (x, y) Directed distance from x-axis

The notation (x, y) denotes both a point in the plane and an open interval on the real number line. The context will tell you which meaning is intended.

Example 1 Plotting Points in the Cartesian Plane

Plot the points $(-1, 2)$, $(3, 4)$, $(0, 0)$, $(3, 0)$, and $(-2, -3)$ in the Cartesian plane.

Solution

To plot the point $(-1, 2)$, imagine a vertical line through -1 on the x-axis and a horizontal line through 2 on the y-axis. The intersection of these two lines is the point $(-1, 2)$. This point is one unit to the left of the y-axis and two units up from the x-axis. The other four points can be plotted in a similar way, as shown in Figure C.3.

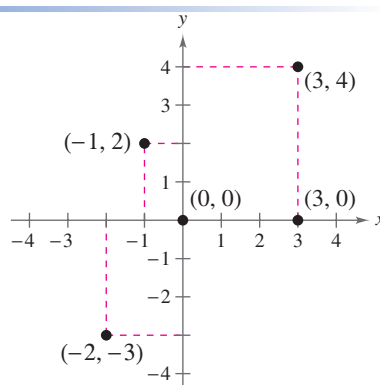


Figure C.3

What you should learn

- Plot points in the Cartesian plane.
- Use the Distance Formula to find the distance between two points.
- Use the Midpoint Formula to find the midpoint of a line segment.
- Find the equation of a circle.
- Translate points in the plane.

Why you should learn it

The Distance Formula can be used to find lengths in real-life situations. For instance, in Exercise 89 on page C9, you will use the Distance Formula to find the length of a football pass.

CHECKPOINT Now try Exercise 13.

The Distance Formula

Recall from the Pythagorean Theorem that, for a right triangle with hypotenuse of length c and sides of lengths a and b , you have

$$a^2 + b^2 = c^2 \quad \text{Pythagorean Theorem}$$

as shown in Figure C.4. (The converse is also true. That is, if $a^2 + b^2 = c^2$, then the triangle is a right triangle.)

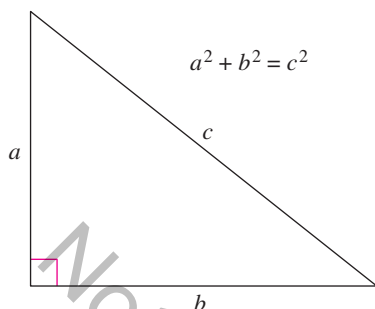


Figure C.4

Suppose you want to determine the distance d between two points (x_1, y_1) and (x_2, y_2) in the plane. With these two points, a right triangle can be formed, as shown in Figure C.5.

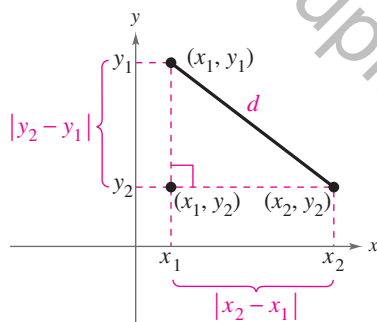


Figure C.5

The length of the vertical side of the triangle is

$$|y_2 - y_1| \quad \text{Length of vertical side}$$

and the length of the horizontal side is

$$|x_2 - x_1|. \quad \text{Length of horizontal side}$$

By the Pythagorean Theorem,

$$\begin{aligned} d^2 &= |x_2 - x_1|^2 + |y_2 - y_1|^2 \\ d &= \sqrt{|x_2 - x_1|^2 + |y_2 - y_1|^2} \\ d &= \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}. \end{aligned}$$

This result is called the **Distance Formula**.

The Distance Formula

The distance d between the points (x_1, y_1) and (x_2, y_2) in the plane is

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}.$$

Example 3 Finding a Distance

Find the distance between the points

$$(-2, 1) \text{ and } (3, 4).$$

Algebraic Solution

Let $(x_1, y_1) = (-2, 1)$ and $(x_2, y_2) = (3, 4)$. Then apply the Distance Formula as follows.

$$\begin{aligned} d &= \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} && \text{Distance Formula} \\ &= \sqrt{[3 - (-2)]^2 + (4 - 1)^2} && \text{Substitute for } x_1, y_1, x_2, \text{ and } y_2. \\ &= \sqrt{(5)^2 + (3)^2} && \text{Simplify.} \\ &= \sqrt{34} && \text{Simplify.} \\ &\approx 5.83 && \text{Use a calculator.} \end{aligned}$$

So, the distance between the points is about 5.83 units.

You can use the Pythagorean Theorem to check that the distance is correct.

$$\begin{aligned} d^2 &\stackrel{?}{=} 3^2 + 5^2 && \text{Pythagorean Theorem} \\ (\sqrt{34})^2 &\stackrel{?}{=} 3^2 + 5^2 && \text{Substitute for } d. \\ 34 &= 34 && \text{Distance checks. } \checkmark \end{aligned}$$

CHECKPOINT Now try Exercise 31.

When the Distance Formula is used, it does not matter which point is (x_1, y_1) and which is (x_2, y_2) , because the result will be the same. For instance, in Example 3, let $(x_1, y_1) = (3, 4)$ and $(x_2, y_2) = (-2, 1)$. Then

$$d = \sqrt{(-2 - 3)^2 + (1 - 4)^2} = \sqrt{(-5)^2 + (-3)^2} = \sqrt{34} \approx 5.83.$$

Example 4 Verifying a Right Triangle

Show that the points

$$(2, 1), (4, 0), \text{ and } (5, 7)$$

are the vertices of a right triangle.

Solution

The three points are plotted in Figure C.7. Using the Distance Formula, you can find the lengths of the three sides as follows.

$$\begin{aligned} d_1 &= \sqrt{(5 - 2)^2 + (7 - 1)^2} = \sqrt{9 + 36} = \sqrt{45} \\ d_2 &= \sqrt{(4 - 2)^2 + (0 - 1)^2} = \sqrt{4 + 1} = \sqrt{5} \\ d_3 &= \sqrt{(5 - 4)^2 + (7 - 0)^2} = \sqrt{1 + 49} = \sqrt{50} \end{aligned}$$

Because

$$(d_1)^2 + (d_2)^2 = 45 + 5 = 50 = (d_3)^2$$

you can conclude that the triangle must be a right triangle.

CHECKPOINT Now try Exercise 45.

Graphical Solution

Use centimeter graph paper to plot the points $A(-2, 1)$ and $B(3, 4)$. Carefully sketch the line segment from A to B . Then use a centimeter ruler to measure the length of the segment.

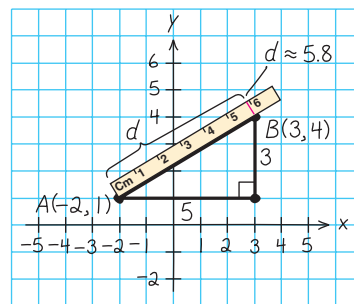


Figure C.6

The line segment measures about 5.8 centimeters, as shown in Figure C.6. So, the distance between the points is about 5.8 units.

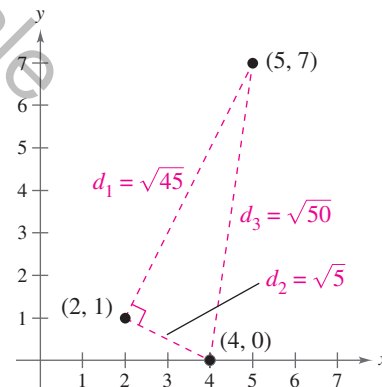


Figure C.7

The Midpoint Formula

To find the **midpoint** of the line segment that joins two points in a coordinate plane, find the average values of the respective coordinates of the two endpoints using the **Midpoint Formula**.

The Midpoint Formula

The midpoint of the line segment joining the points (x_1, y_1) and (x_2, y_2) is given by the Midpoint Formula

$$\text{Midpoint} = \left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2} \right).$$

Example 5 Finding a Line Segment's Midpoint

Find the midpoint of the line segment joining the points $(-5, -3)$ and $(9, 3)$.

Solution

Let $(x_1, y_1) = (-5, -3)$ and $(x_2, y_2) = (9, 3)$:

$$\begin{aligned} \text{Midpoint} &= \left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2} \right) && \text{Midpoint Formula} \\ &= \left(\frac{-5 + 9}{2}, \frac{-3 + 3}{2} \right) && \text{Substitute for } x_1, y_1, x_2, \text{ and } y_2. \\ &= (2, 0) && \text{Simplify.} \end{aligned}$$

The midpoint of the line segment is $(2, 0)$, as shown in Figure C.8.

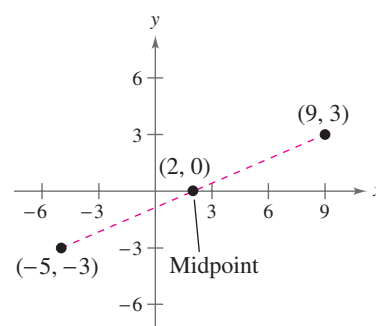


Figure C.8

CHECKPOINT Now try Exercise 57.

Example 6 Estimating Annual Revenues



Verizon Communications had annual revenues of \$88.1 billion in 2006 and \$97.4 billion in 2008. Without knowing any additional information, what would you estimate the 2007 revenue to have been? (Source: Verizon Communications)

Solution

One solution to the problem is to assume that revenue followed a linear pattern. With this assumption, you can estimate the 2007 revenue by finding the midpoint of the line segment connecting the points $(2006, 88.1)$ and $(2008, 97.4)$.

$$\begin{aligned} \text{Midpoint} &= \left(\frac{2006 + 2008}{2}, \frac{88.1 + 97.4}{2} \right) \\ &\approx (2007, 92.8) \end{aligned}$$

So, you would estimate the 2007 revenue to have been about \$92.8 billion, as shown in Figure C.9. (The actual 2007 revenue was \$93.5 billion.)

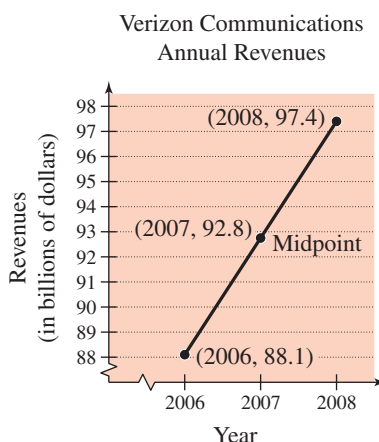


Figure C.9

CHECKPOINT Now try Exercise 63.

The Equation of a Circle

The Distance Formula provides a convenient way to define circles. A **circle of radius r** with center at the point (h, k) is shown in Figure C.10. The point (x, y) is on this circle if and only if its distance from the center (h, k) is r . This means that a **circle** in the plane consists of all points (x, y) that are a given positive distance r from a fixed point (h, k) . Using the Distance Formula, you can express this relationship by saying that the point (x, y) lies on the circle if and only if

$$\sqrt{(x - h)^2 + (y - k)^2} = r.$$

By squaring each side of this equation, you obtain the **standard form of the equation of a circle**.

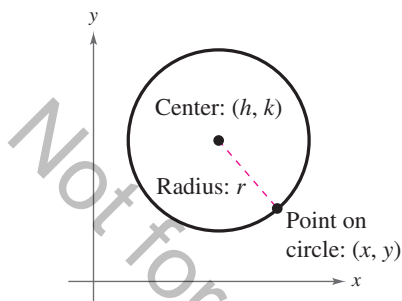


Figure C.10

Standard Form of the Equation of a Circle

The **standard form of the equation of a circle** is

$$(x - h)^2 + (y - k)^2 = r^2.$$

The point (h, k) is the **center** of the circle, and the positive number r is the **radius** of the circle. The standard form of the equation of a circle whose center is the origin, $(h, k) = (0, 0)$, is $x^2 + y^2 = r^2$.

Example 7 Writing an Equation of a Circle

The point $(3, 4)$ lies on a circle whose center is at $(-1, 2)$, as shown in Figure C.11. Write the standard form of the equation of this circle.

Solution

The radius r of the circle is the distance between $(-1, 2)$ and $(3, 4)$.

$$\begin{aligned} r &= \sqrt{[3 - (-1)]^2 + (4 - 2)^2} && \text{Substitute for } x, y, h, \text{ and } k. \\ &= \sqrt{4^2 + 2^2} && \text{Simplify.} \\ &= \sqrt{16 + 4} && \text{Simplify.} \\ &= \sqrt{20} && \text{Radius} \end{aligned}$$

Using $(h, k) = (-1, 2)$ and $r = \sqrt{20}$, the equation of the circle is

$$\begin{aligned} (x - h)^2 + (y - k)^2 &= r^2 && \text{Equation of circle} \\ [x - (-1)]^2 + (y - 2)^2 &= (\sqrt{20})^2 && \text{Substitute for } h, k, \text{ and } r. \\ (x + 1)^2 + (y - 2)^2 &= 20. && \text{Standard form} \end{aligned}$$

CHECKPOINT Now try Exercise 67.

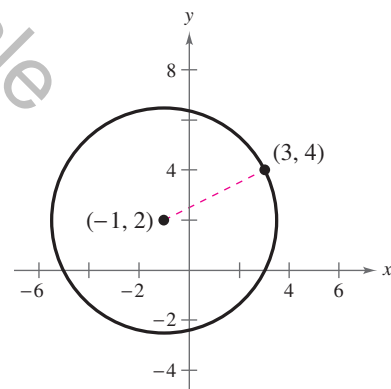


Figure C.11

Application

Much of computer graphics consists of transformations of points in a coordinate plane. One type of transformation, a translation, is illustrated in Example 8. Other types of transformations include reflections, rotations, and stretches.

Example 8 Translating Points in the Plane

The triangle in Figure C.12 has vertices at the points

$(-1, 2)$, $(1, -4)$, and $(2, 3)$.

Shift the triangle three units to the right and two units upward and find the vertices of the shifted triangle, as shown in Figure C.13.

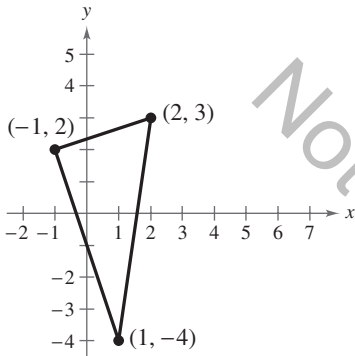


Figure C.12

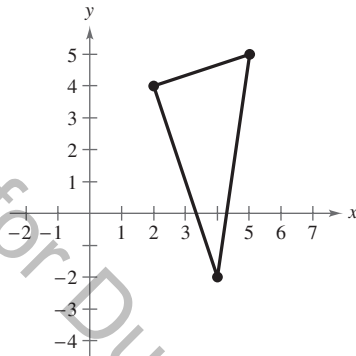


Figure C.13

Solution

To shift the vertices three units to the right, add 3 to each of the x -coordinates. To shift the vertices two units upward, add 2 to each of the y -coordinates.

Original Point	Translated Point
$(-1, 2)$	$(-1 + 3, 2 + 2) = (2, 4)$
$(1, -4)$	$(1 + 3, -4 + 2) = (4, -2)$
$(2, 3)$	$(2 + 3, 3 + 2) = (5, 5)$

Plotting the translated points and sketching the line segments between them produces the shifted triangle shown in Figure C.13.

CHECKPOINT Now try Exercise 85.

Example 8 shows how to translate points in a coordinate plane. The following transformed points are related to the original points as follows.

Original Point	Transformed Point	
(x, y)	$(-x, y)$	$(-x, y)$ is a reflection of the original point in the y -axis.
(x, y)	$(x, -y)$	$(x, -y)$ is a reflection of the original point in the x -axis.
(x, y)	$(-x, -y)$	$(-x, -y)$ is a reflection of the original point through the origin.

The figures provided with Example 8 were not really essential to the solution. Nevertheless, it is strongly recommended that you develop the habit of including sketches with your solutions because they serve as useful problem-solving tools.

C.1 Exercises

For instructions on how to use a graphing utility, see Appendix A.

Vocabulary and Concept Check

In Exercises 1–4, fill in the blank(s).

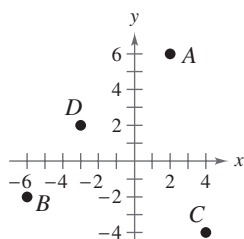
1. An ordered pair of real numbers can be represented in a plane called the rectangular coordinate system or the _____ plane.
2. The _____ is a result derived from the Pythagorean Theorem.
3. Finding the average values of the respective coordinates of the two endpoints of a line segment in a coordinate plane is also known as using the _____.
4. The standard form of the equation of a circle is _____, where the point (h, k) is the _____ of the circle and the positive number r is the _____ of the circle.

In Exercises 5–10, match each term with its definition.

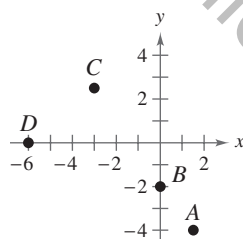
- | | |
|---------------------|--|
| 5. x -axis | (a) point of intersection of vertical axis and horizontal axis |
| 6. y -axis | (b) directed distance from the x -axis |
| 7. origin | (c) horizontal real number line |
| 8. quadrants | (d) four regions of the coordinate plane |
| 9. x -coordinate | (e) directed distance from the y -axis |
| 10. y -coordinate | (f) vertical real number line |

Procedures and Problem Solving**Approximating Coordinates of Points** In Exercises 11 and 12, approximate the coordinates of the points.

11.



12.

**Plotting Points in the Cartesian Plane** In Exercises 13–16, plot the points in the Cartesian plane.

- ✓ 13. $(-4, 2)$, $(-3, -6)$, $(0, 5)$, $(1, -4)$
 14. $(4, -2)$, $(0, 0)$, $(-4, 0)$, $(-5, -5)$
 15. $(3, 8)$, $(0.5, -1)$, $(5, -6)$, $(-2, -2.5)$
 16. $(1, -\frac{1}{2})$, $(-\frac{5}{2}, 2)$, $(3, -3)$, $(\frac{3}{2}, 1)$

Finding Coordinates of Points In Exercises 17–20, find the coordinates of the point.

17. The point is located five units to the left of the y -axis and four units above the x -axis.
18. The point is located three units below the x -axis and two units to the right of the y -axis.
19. The point is on the y -axis and six units below the x -axis.
20. The point is on the x -axis and 11 units to the left of the y -axis.

Determining Quadrants In Exercises 21–30, determine the quadrant(s) in which (x, y) is located so that the condition(s) is (are) satisfied.

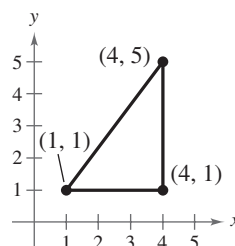
- | | |
|--------------------------|--------------------------|
| 21. $x > 0$ and $y < 0$ | 22. $x < 0$ and $y < 0$ |
| 23. $x = -4$ and $y > 0$ | 24. $x > 2$ and $y = 3$ |
| 25. $y < -5$ | 26. $x > 4$ |
| 27. $x < 0$ and $-y > 0$ | 28. $-x > 0$ and $y < 0$ |
| 29. $xy > 0$ | 30. $xy < 0$ |

Finding a Distance In Exercises 31–40, find the distance between the points algebraically and confirm graphically by using centimeter graph paper and a centimeter ruler.

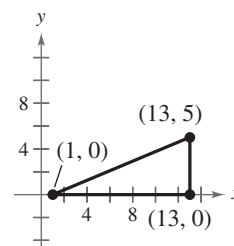
- ✓ 31. $(6, -3)$, $(6, 5)$ 32. $(-3, 0)$, $(-3, -7)$
 33. $(-3, -1)$, $(2, -1)$ 34. $(-11, 4)$, $(-1, 4)$
 35. $(-2, 6)$, $(3, -6)$ 36. $(8, 5)$, $(0, 20)$
 37. $(\frac{1}{2}, \frac{4}{3})$, $(2, -1)$ 38. $(-\frac{2}{3}, 3)$, $(-1, \frac{5}{4})$
 39. $(-4.2, 3.1)$, $(-12.5, 4.8)$ 40. $(9.5, -2.6)$, $(-3.9, 8.2)$

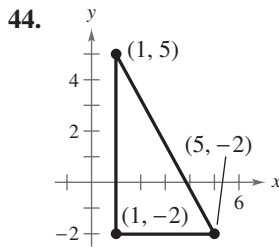
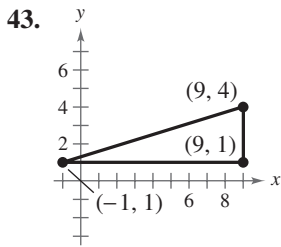
Verifying a Right Triangle In Exercises 41–44, (a) find the length of each side of the right triangle and (b) show that these lengths satisfy the Pythagorean Theorem.

41.



42.





Verifying a Polygon In Exercises 45–52, show that the points form the vertices of the polygon.


- ✓ 45. Right triangle: (4, 0), (2, 1), (−1, −5)
 46. Right triangle: (−1, 3), (3, 5), (5, 1)
 47. Isosceles triangle: (1, −3), (3, 2), (−2, 4)
 48. Isosceles triangle: (2, 3), (4, 9), (−2, 7)
 49. Parallelogram: (2, 5), (0, 9), (−2, 0), (0, −4)
 50. Parallelogram: (0, 1), (3, 7), (4, 4), (1, −2)
 51. Rectangle: (−5, 6), (0, 8), (−3, 1), (2, 3) (*Hint*: Show that the diagonals are of equal length.)
 52. Rectangle: (2, 4), (3, 1), (1, 2), (4, 3) (*Hint*: Show that the diagonals are of equal length.)

Finding a Line Segment's Midpoint In Exercises 53–62, (a) plot the points, (b) find the distance between the points, and (c) find the midpoint of the line segment joining the points.


53. (0, 0), (8, 6) 54. (1, 12), (6, 0)
 55. (−4, 10), (4, −5) 56. (−7, −4), (2, 8)
 ✓ 57. (−1, 2), (5, 4) 58. (2, 10), (10, 2)
 59. $(\frac{1}{2}, 1)$, $(-\frac{5}{2}, \frac{4}{3})$ 60. $(-\frac{1}{3}, -\frac{1}{3})$, $(-\frac{1}{6}, -\frac{1}{2})$
 61. (6.2, 5.4), (−3.7, 1.8) 62. (−16.8, 12.3), (5.6, 4.9)

Estimating Annual Revenues In Exercises 63 and 64, use the Midpoint Formula to estimate the annual revenues (in millions of dollars) for Texas Roadhouse and Papa John's in 2006. The revenues for the two companies in 2003 and 2009 are shown in the tables. Assume that the revenues followed a linear pattern. (Sources: Texas Roadhouse, Inc.; Papa John's International)

- ✓ 63. Texas Roadhouse

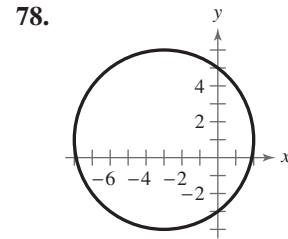
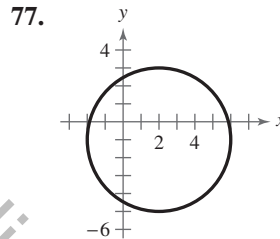
	Year	Annual revenue (in millions of dollars)
	2003	287
	2009	942

64. Papa John's Intl.

	Year	Annual revenue (in millions of dollars)
	2003	917
	2009	1106

Writing an Equation of a Circle In Exercises 65–78, write the standard form of the equation of the specified circle.

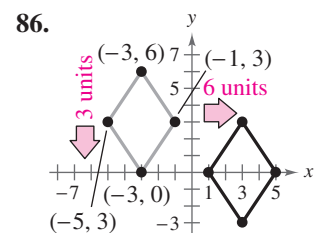
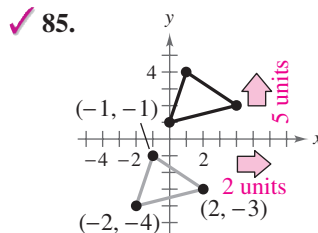
65. Center: (0, 0); radius: 5
 66. Center: (0, 0); radius: 6
 ✓ 67. Center: (2, −1); radius: 4
 68. Center: (−5, 3); radius: 2
 69. Center: (−1, 2); solution point: (0, 0)
 70. Center: (3, −2); solution point: (−1, 1)
 71. Endpoints of a diameter: (0, 0), (6, 8)
 72. Endpoints of a diameter: (−4, −1), (4, 1)
 73. Center: (−2, 1); tangent to the x-axis
 74. Center: (3, −2); tangent to the y-axis
 75. The circle inscribed in the square with vertices (7, −2), (−1, −2), (−1, −10), and (7, −10)
 76. The circle inscribed in the square with vertices (−12, 10), (8, 10), (8, −10), and (−12, −10)



Sketching a Circle In Exercises 79–84, find the center and radius, and sketch the circle.

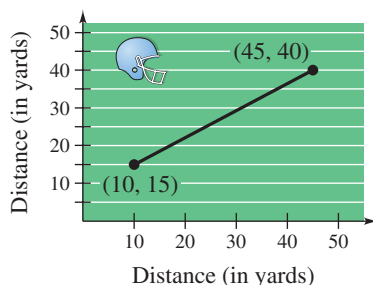
79. $x^2 + y^2 = 9$ 80. $x^2 + y^2 = 16$
 81. $(x - 1)^2 + (y + 3)^2 = 4$
 82. $x^2 + (y - 1)^2 = 49$ 83. $(x - \frac{1}{2})^2 + (y - \frac{1}{2})^2 = \frac{9}{4}$
 84. $(x - \frac{2}{3})^2 + (y + \frac{1}{4})^2 = \frac{25}{36}$

Translating Points in the Plane In Exercises 85–88, the polygon is shifted to a new position in the plane. Find the coordinates of the vertices of the polygon in the new position.

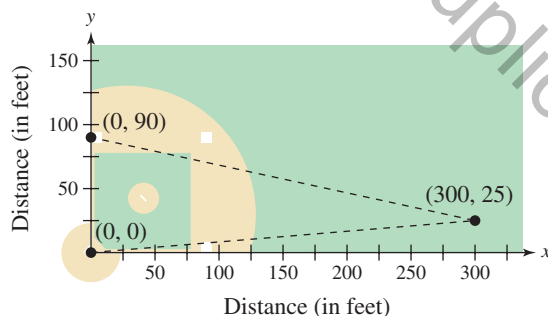


87. Original coordinates of vertices:
 (0, 2), (−3, 5), (−5, 2), (−2, −1)
 Shift: three units upward, one unit to the left
 88. Original coordinates of vertices:
 (1, −1), (3, 2), (1, −2)
 Shift: two units downward, three units to the left

89. **Why you should learn it** (p. C1) In a football game, a quarterback throws a pass from the 15-yard line, 10 yards from the sideline, as shown in the figure. The pass is caught on the 40-yard line, 45 yards from the same sideline. How long is the pass?



90. **Physical Education** A major league baseball diamond is a square with 90-foot sides. In the figure, home plate is at the origin and the first base line lies on the positive x -axis. The right fielder fields the ball at the point $(300, 25)$. How far does the right fielder have to throw the ball to get a runner out at home plate? How far does the right fielder have to throw the ball to get a runner out at third base? (Round your answers to one decimal place.)



91. **Aviation** A jet plane flies from Naples, Italy in a straight line to Rome, Italy, which is 120 kilometers north and 150 kilometers west of Naples. How far does the plane fly?

Conclusions

True or False? In Exercises 92–94, determine whether the statement is true or false. Justify your answer.

92. The points $(-8, 4)$, $(2, 11)$, and $(-5, 1)$ represent the vertices of an isosceles triangle.
93. If four points represent the vertices of a polygon, and the four sides are equal, then the polygon must be a square.
94. In order to divide a line segment into 16 equal parts, you would have to use the Midpoint Formula 16 times.
95. **Think About It** What is the y -coordinate of any point on the x -axis? What is the x -coordinate of any point on the y -axis?

96. **Think About It** When plotting points on the rectangular coordinate system, is it true that the scales on the x - and y -axes must be the same? Explain.

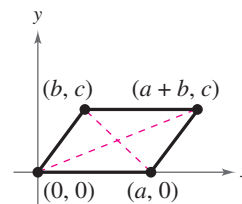
97. **Exploration** A line segment has (x_1, y_1) as one endpoint and (x_m, y_m) as its midpoint. Find the other endpoint (x_2, y_2) of the line segment in terms of x_1 , y_1 , x_m , and y_m . Use the result to find the coordinates of the endpoint of a line segment when the coordinates of the other endpoint and midpoint are, respectively,

(a) $(1, -2)$, $(4, -1)$. (b) $(-5, 11)$, $(2, 4)$.

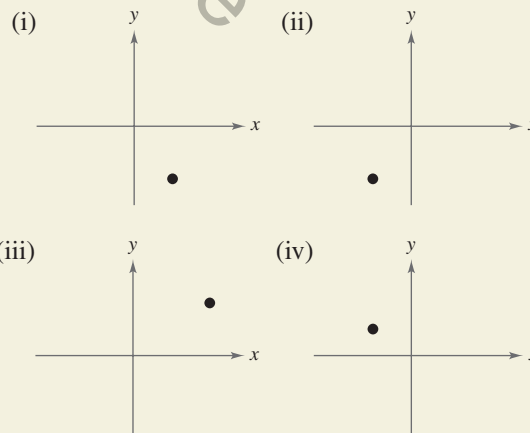
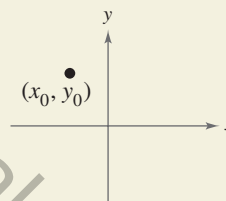
98. **Exploration** Use the Midpoint Formula three times to find the three points that divide the line segment joining (x_1, y_1) and (x_2, y_2) into four parts. Use the result to find the points that divide the line segment joining the given points into four equal parts.

(a) $(1, -2)$, $(4, -1)$ (b) $(-2, -3)$, $(0, 0)$

99. **Proof** Prove that the diagonals of the parallelogram in the figure intersect at their midpoints.



100. **CAPSTONE** Use the plot of the point (x_0, y_0) in the figure. Match the transformation of the point with the correct plot. Explain your reasoning. [The plots are labeled (i), (ii), (iii), and (iv).]



- (a) $(x_0, -y_0)$ (b) $(-2x_0, y_0)$
 (c) $(x_0, \frac{1}{2}y_0)$ (d) $(-x_0, -y_0)$

C.2 Graphs of Equations

The Graph of an Equation

News magazines often show graphs comparing the rate of inflation, the federal deficit, or the unemployment rate to the time of year. Businesses use graphs to report monthly sales statistics. Such graphs provide geometric pictures of the way one quantity changes with respect to another. Frequently, the relationship between two quantities is expressed as an **equation**. This appendix introduces the basic procedure for determining the geometric picture associated with an equation.

For an equation in the variables x and y , a point (a, b) is a **solution point** when substitution of a for x and b for y satisfies the equation. Most equations have *infinitely many* solution points. For example, the equation

$$3x + y = 5$$

has solution points

$$(0, 5), (1, 2), (2, -1), (3, -4)$$

and so on. The set of all solution points of an equation is the **graph of the equation**.

Example 1 Determining Solution Points

Determine whether (a) $(2, 13)$ and (b) $(-1, -3)$ lie on the graph of

$$y = 10x - 7.$$

Solution

a. $y = 10x - 7$ Write original equation.
 $13 \stackrel{?}{=} 10(2) - 7$ Substitute 2 for x and 13 for y .
 $13 = 13$ $(2, 13)$ is a solution. ✓

The point $(2, 13)$ *does* lie on the graph of $y = 10x - 7$ because it is a solution point of the equation.

b. $y = 10x - 7$ Write original equation.
 $-3 \stackrel{?}{=} 10(-1) - 7$ Substitute -1 for x and -3 for y .
 $-3 \neq -17$ $(-1, -3)$ is not a solution.

The point $(-1, -3)$ *does not* lie on the graph of $y = 10x - 7$ because it is not a solution point of the equation.

 **CHECKPOINT** Now try Exercise 7.

The basic technique used for sketching the graph of an equation is the point-plotting method.

What you should learn

- Sketch graphs of equations by point plotting.
- Graph equations using a graphing utility.
- Use graphs of equations to solve real-life problems.

Why you should learn it

The graph of an equation can help you see relationships between real-life quantities. For example, in Exercise 66 on page C18, a graph can be used to estimate the life expectancies of children born in the years 1948 and 2020.

Sketching the Graph of an Equation by Point Plotting

1. If possible, rewrite the equation so that one of the variables is isolated on one side of the equation.
2. Make a table of values showing several solution points.
3. Plot these points on a rectangular coordinate system.
4. Connect the points with a smooth curve or line.

Example 2 Sketching a Graph by Point Plotting

Use point plotting and graph paper to sketch the graph of $3x + y = 6$.

Solution

In this case you can isolate the variable y .

$$y = 6 - 3x \quad \text{Solve equation for } y.$$

Using negative and positive values of x , and $x = 0$, you can obtain the following table of values (solution points).

x	-1	0	1	2	3
$y = 6 - 3x$	9	6	3	0	-3
Solution point	$(-1, 9)$	$(0, 6)$	$(1, 3)$	$(2, 0)$	$(3, -3)$

Next, plot the solution points and connect them, as shown in Figure C.14. It appears that the graph is a straight line. You will study lines extensively in Section 1.1.

CHECKPOINT Now try Exercise 11.

The points at which a graph touches or crosses an axis are called the **intercepts** of the graph. For instance, in Example 2 the point

$$(0, 6) \quad \text{y-intercept}$$

is the y -intercept of the graph because the graph crosses the y -axis at that point. The point

$$(2, 0) \quad \text{x-intercept}$$

is the x -intercept of the graph because the graph crosses the x -axis at that point.

Example 3 Sketching a Graph by Point Plotting

Use point plotting and graph paper to sketch the graph of

$$y = x^2 - 2.$$

Solution

Because the equation is already solved for y , make a table of values by choosing several convenient values of x and calculating the corresponding values of y .

x	-2	-1	0	1	2	3
$y = x^2 - 2$	2	-1	-2	-1	2	7
Solution point	$(-2, 2)$	$(-1, -1)$	$(0, -2)$	$(1, -1)$	$(2, 2)$	$(3, 7)$

Next, plot the solution points, as shown in Figure C.15(a). Finally, connect the points with a smooth curve, as shown in Figure C.15(b). This graph is called a *parabola*. You will study parabolas in Section 2.1.

CHECKPOINT Now try Exercise 13.

In this text, you will study two basic ways to create graphs: *by hand* and *using a graphing utility*. For instance, the graphs in Figures C.14 and C.15 were sketched by hand, and the graph in Figure C.18 (on the next page) was created using a graphing utility.

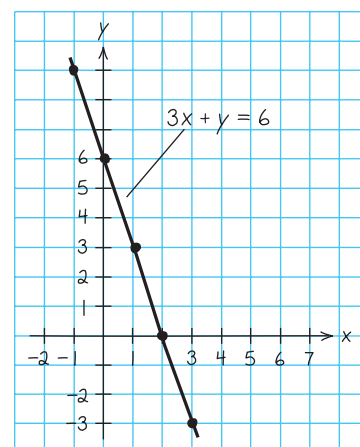
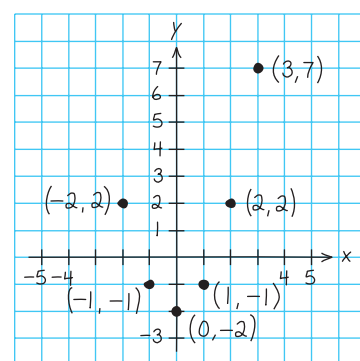
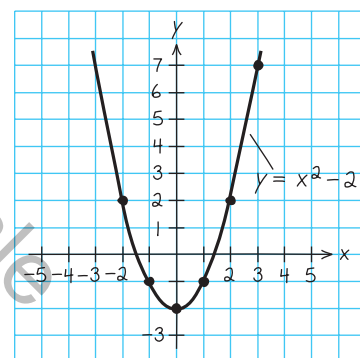


Figure C.14



(a)



(b)

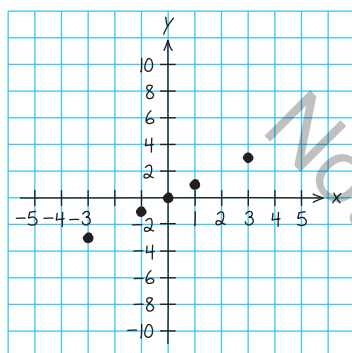
Figure C.15

Using a Graphing Utility

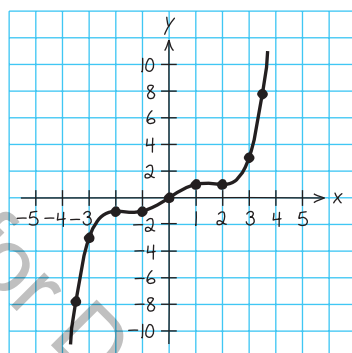
One of the disadvantages of the point-plotting method is that to get a good idea about the shape of a graph, you need to plot *many* points. With only a few points, you could misrepresent the graph of an equation. For instance, consider the equation

$$y = \frac{1}{30}x(x^4 - 10x^2 + 39).$$

When you plot the points $(-3, -3)$, $(-1, -1)$, $(0, 0)$, $(1, 1)$, and $(3, 3)$, as shown in Figure C.16(a), you might think that the graph of the equation is a line. This is not correct. By plotting several more points and connecting the points with a smooth curve, you can see that the actual graph is not a line, as shown in Figure C.16(b).



(a)
Figure C.16



(b)

From this, you can see that the point-plotting method leaves you with a dilemma. This method can be very inaccurate when only a few points are plotted, and it is very time-consuming to plot a dozen (or more) points. Technology can help solve this dilemma. Plotting several (even several hundred) points on a rectangular coordinate system is something that a computer or calculator can do easily. For instance, you can enter the equation $y = \frac{1}{30}x(x^4 - 10x^2 + 39)$ in a graphing utility (see Figure C.17) to obtain the graph shown in Figure C.18.

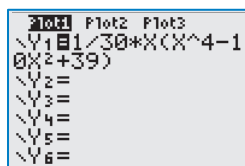


Figure C.17

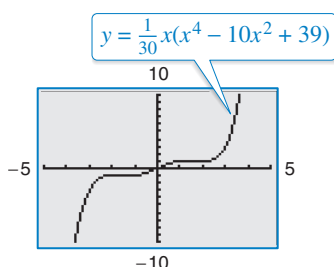


Figure C.18

Technology Tip



Many graphing utilities are capable of creating a table of values such as the following, which shows some points of the graph in Figure C.16(b). For instructions on how to use the *table* feature, see Appendix A; for specific keystrokes, go to this textbook's *Companion Website*.

X	Y1
-3	-1
-1	-1
0	0
1	1
3	1
18	18

Technology Tip



By choosing different viewing windows for a graph, it is possible to obtain very different impressions of the graph's shape. For instance, Figure C.19 shows a different viewing window for the graph of the equation in Figure C.18. Note how Figure C.19 does not show *all* of the important features of the graph as does Figure C.18. For instructions on how to set up a viewing window, see Appendix A; for specific keystrokes, go to this textbook's *Companion Website*.

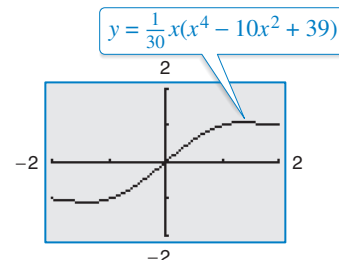


Figure C.19

Using a Graphing Utility to Graph an Equation

To graph an equation involving x and y on a graphing utility, do the following.

1. Rewrite the equation so that y is isolated on the left side.
2. Enter the equation in the graphing utility.
3. Determine a *viewing window* that shows all important features of the graph.
4. Graph the equation.

Example 4 Using a Graphing Utility to Graph an Equation

To graph

$$y = -\frac{x^3}{2} + 2x$$

enter the equation in a graphing utility. Then use a standard viewing window (see Figure C.20) to obtain the graph shown in Figure C.21.

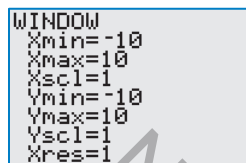


Figure C.20

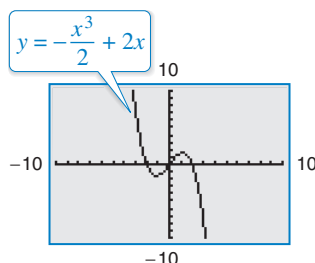


Figure C.21

CHECKPOINT Now try Exercise 43.

Example 5 Using a Graphing Utility to Graph a Circle

Use a graphing utility to graph $x^2 + y^2 = 9$.

Solution

The graph of $x^2 + y^2 = 9$ is a circle whose center is the origin and whose radius is 3. To graph the equation, begin by solving the equation for y .

$$x^2 + y^2 = 9$$

Write original equation.

$$y^2 = 9 - x^2$$

Subtract x^2 from each side.

$$y = \pm\sqrt{9 - x^2}$$

Take the square root of each side.

Remember that when you take the square root of a variable expression, you must account for both the positive and negative solutions. The graph of $y = \sqrt{9 - x^2}$ is the upper semicircle. The graph of $y = -\sqrt{9 - x^2}$ is the lower semicircle. Enter *both* equations in your graphing utility and generate the resulting graphs. In Figure C.22, note that for a standard viewing window, the two graphs do not appear to form a circle. You can overcome this problem by using a *square setting*, in which the horizontal and vertical tick marks have equal spacing, as shown in Figure C.23. On many graphing utilities, a square setting can be obtained by using a y to x ratio of 2 to 3. For instance, in Figure C.23, the y to x ratio is

$$\frac{Y_{\max} - Y_{\min}}{X_{\max} - X_{\min}} = \frac{4 - (-4)}{6 - (-6)} = \frac{8}{12} = \frac{2}{3}.$$

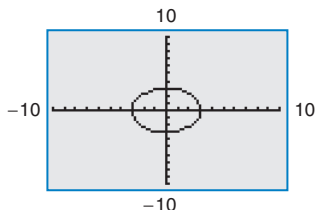


Figure C.22

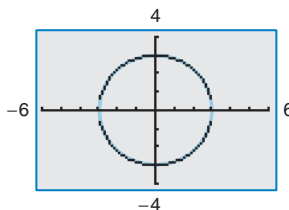


Figure C.23

CHECKPOINT Now try Exercise 57.

Technology Tip

The standard viewing window on many graphing utilities does not give a true geometric perspective because the screen is rectangular, which distorts the image. That is, perpendicular lines will not appear to be perpendicular and circles will not appear to be circular. To overcome this, you can use a *square setting*, as demonstrated in Example 5.

Technology Tip

Notice that when you graph a circle by graphing two separate equations for y , your graphing utility may not connect the two semicircles. This is because some graphing utilities are limited in their resolution. So, in this text, a blue curve is placed behind the graphing utility's display to indicate where the graph should appear.

Applications

Throughout this course, you will learn that there are many ways to approach a problem. Two of the three common approaches are illustrated in Example 6.

An Algebraic Approach: Use the rules of algebra.

A Graphical Approach: Draw and use a graph.

A Numerical Approach: Construct and use a table.

You should develop the habit of using at least two approaches to solve every problem in order to build your intuition and to check that your answer is reasonable.

The following two applications show how to develop mathematical models to represent real-world situations. You will see that both a graphing utility and algebra can be used to understand and solve the problems posed.

Example 6 Running a Marathon



A runner runs at a constant rate of 4.9 miles per hour. The verbal model and algebraic equation relating distance run and elapsed time are as follows.

Verbal Model: Distance = Rate · Time

Equation: $d = 4.9t$

- Determine how far the runner can run in 3.1 hours.
- Determine how long it will take the runner to run a 26.2-mile marathon.

Algebraic Solution

- To begin, find how far the runner can run in 3.1 hours by substituting 3.1 for t in the equation.

$$\begin{aligned} d &= 4.9t && \text{Write original equation.} \\ &= 4.9(3.1) && \text{Substitute 3.1 for } t. \\ &\approx 15.2 && \text{Use a calculator.} \end{aligned}$$

So, the runner can run about 15.2 miles in 3.1 hours. Use estimation to check your answer. Because 4.9 is about 5 and 3.1 is about 3, the distance is about

$$5(3) = 15.$$

So, 15.2 is reasonable.

- You can find how long it will take to run a 26.2-mile marathon as follows. (For help with solving linear equations, see Appendix E at this textbook's *Companion Website*.)

$$\begin{aligned} d &= 4.9t && \text{Write original equation.} \\ 26.2 &= 4.9t && \text{Substitute 26.2 for } d. \\ \frac{26.2}{4.9} &= t && \text{Divide each side by 4.9.} \\ 5.3 &\approx t && \text{Use a calculator.} \end{aligned}$$

So, it will take the runner about 5.3 hours to run 26.2 miles.

Graphical Solution

- Use a graphing utility to graph $d = 4.9t$. (Represent d by y and t by x .) Choose a viewing window that shows the graph at $x = 3.1$.

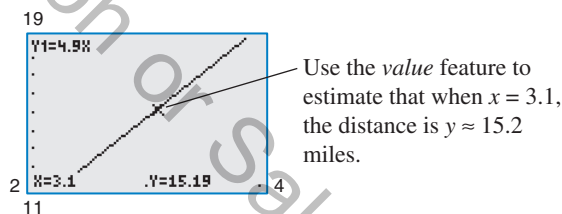


Figure C.24

- Adjust the viewing window so that it shows the graph at $y = 26.2$.

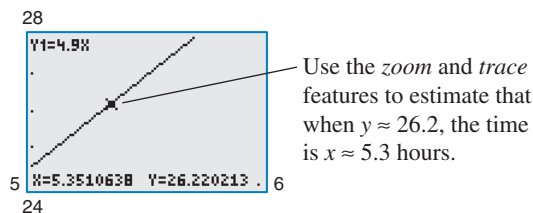


Figure C.25

Note that the viewing window on your graphing utility may differ slightly from those shown in Figures C.24 and C.25.

CHECKPOINT Now try Exercise 63.

Example 7 Monthly Wage

You receive a monthly salary of \$2000 plus a commission of 10% of sales. The verbal model and algebraic equation relating the wages, the salary, and the commission are as follows.

Verbal Model: **Wages** = **Salary** + **Commission on sales**

Equation: $y = 2000 + 0.1x$

- Sales are \$1480 in August. What are your wages for that month?
- You receive \$2225 for September. What are your sales for that month?

Numerical Solution

- Enter $y = 2000 + 0.1x$ in a graphing utility. Then use the *table* feature of the graphing utility to create a table. Start the table at $x = 1400$ with a table step of 10.

X	Y ₁
1420	2142
1430	2143
1440	2144
1450	2145
1460	2146
1470	2147
1480	2148

When $x = 1480$, the wages are $y = \$2148$.

- Adjust the table to start at $x = 2000$ with a table step of 100.

X	Y ₁
2000	2200
2100	2210
2200	2220
2300	2230
2400	2240
2500	2250
2600	2260

From the table, you can see that wages of \$2225 result from sales between \$2200 and \$2300.

You can improve the estimate by starting the table at $x = 2200$ with a table step of 10.

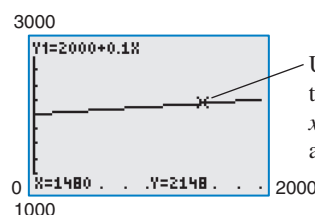
From the table, you can see that wages of \$2225 result from sales of \$2250.

X	Y ₁
2200	2220
2210	2221
2220	2222
2230	2223
2240	2224
2250	2225
2260	2226

CHECKPOINT Now try Exercise 65.

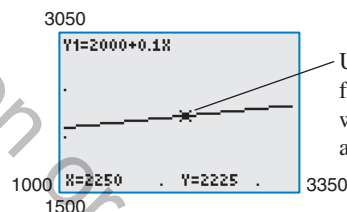
Graphical Solution

- Use a graphing utility to graph $y = 2000 + 0.1x$. Choose a viewing window that shows the graph at $x = 1480$.



Use the *value* feature to estimate that when $x = 1480$, the wages are $y = \$2148$.

- Use the graphing utility to find the value along the x -axis (sales) that corresponds to a y -value of 2225 (wages). Adjust the viewing window so that it shows the graph at $y = 2225$.



Use the *zoom* and *trace* features to estimate that when $y = 2225$, the sales are $x = \$2250$.

Remember to use a different approach to check that your answer is reasonable. For instance, to check the numerical solution to Example 7, use a graphical approach as shown above or use an algebraic approach as follows.

- Substitute 1480 for x in the original equation and solve for y .

$$y = 2000 + 0.1(1480) = \$2148$$

- Substitute 2225 for y in the original equation and solve for x .

$$2225 = 2000 + 0.1x \quad \Rightarrow \quad x = \$2250$$

C.2 Exercises

For instructions on how to use a graphing utility, see Appendix A.

Vocabulary and Concept Check

In Exercises 1 and 2, fill in the blank.

- For an equation in x and y , if substitution of a for x and b for y satisfies the equation, then the point (a, b) is a _____.
- The set of all solution points of an equation is the _____ of the equation.
- Name three common approaches you can use to solve problems mathematically.
- List the steps for sketching the graph of an equation by point plotting.

Procedures and Problem Solving

Determining Solution Points In Exercises 5–10, determine whether each point lies on the graph of the equation.

Equation	Points	
5. $y = \sqrt{x+4}$	(a) (0, 2)	(b) (12, 4)
6. $y = x^2 - 3x + 2$	(a) (2, 0)	(b) (-2, 8)
✓ 7. $y = 4 - x - 2 $	(a) (1, 5)	(b) (1.2, 3.2)
8. $2x - y - 3 = 0$	(a) (1, 2)	(b) (1, -1)
9. $x^2 + y^2 = 20$	(a) (3, -2)	(b) (-4, 2)
10. $y = \frac{1}{3}x^3 - 2x^2$	(a) $(2, -\frac{16}{3})$	(b) (-3, 9)

Sketching a Graph by Point Plotting In Exercises 11–14, complete the table. Use the resulting solution points to sketch the graph of the equation. Use a graphing utility to verify the graph.

✓ 11. $3x - 2y = 2$

x	-2	0	$\frac{2}{3}$	1	2
y					
Solution point					

12. $8x + 4y = 24$

x	-3	-1	0	2	3
y					
Solution point					

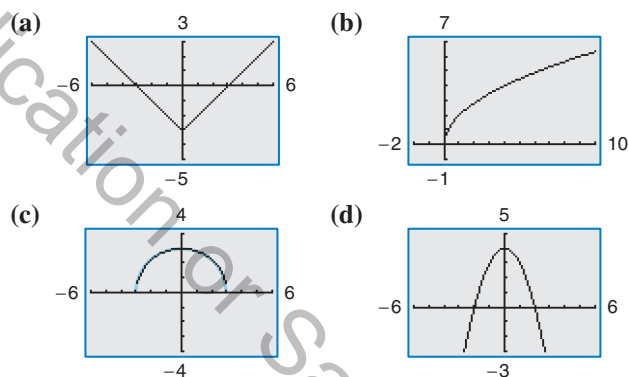
✓ 13. $2x + y = x^2$

x	-1	0	1	2	3
y					
Solution point					

14. $6x - 2y = -2x^2$

x	-4	-3	-2	0	1
y					
Solution point					

Matching an Equation with Its Graph In Exercises 15–18, match the equation with its graph. [The graphs are labeled (a), (b), (c), and (d).]



15. $y = 2\sqrt{x}$

16. $y = 4 - x^2$

17. $y = \sqrt{9 - x^2}$

18. $y = |x| - 3$

Sketching the Graph of an Equation In Exercises 19–32, sketch the graph of the equation.

19. $y = -4x + 1$

20. $y = 2x - 3$

21. $y = 2 - x^2$

22. $y = x^2 - 1$

23. $y = x^2 - 3x$

24. $y = -x^2 - 4x$

25. $y = x^3 + 2$

26. $y = x^3 - 3$

27. $y = \sqrt{x - 3}$

28. $y = \sqrt{1 - x}$

29. $y = |x - 2|$

30. $y = 4 - |x|$

31. $x = y^2 - 1$

32. $x = y^2 + 4$

Using a Graphing Utility to Graph an Equation In Exercises 33–46, use a graphing utility to graph the equation. Use a standard viewing window. Approximate any x - or y -intercepts of the graph.

33. $y = x - 7$

34. $y = x + 1$

35. $y = 3 - \frac{1}{2}x$

36. $y = \frac{2}{3}x - 1$

37. $y = \frac{2x}{x-1}$

38. $y = \frac{6}{x}$

39. $y = x\sqrt{x+3}$

40. $y = (6-x)\sqrt{x}$

41. $y = \sqrt[3]{x-8}$

42. $y = \sqrt[3]{x+1}$

✓ 43. $x^2 - y = 4x - 3$

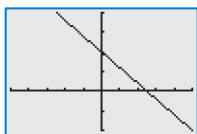
44. $2y - x^2 + 8 = 2x$

45. $y - 4x = x^2(x - 4)$

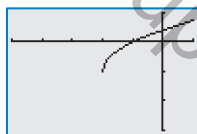
46. $x^3 + y = 1$

Describing the Viewing Window of a Graphing Utility In Exercises 47 and 48, describe the viewing window of the graph shown.

47. $y = -10x + 50$



48. $y = \sqrt{x+2} - 1$



Verifying a Rule of Algebra In Exercises 49–52, explain how to use a graphing utility to verify that $y_1 = y_2$. Identify the rule of algebra that is illustrated.

49. $y_1 = \frac{1}{4}(x^2 - 8)$

50. $y_1 = \frac{1}{2}x + (x + 1)$

$y_2 = \frac{1}{4}x^2 - 2$

$y_2 = \frac{3}{2}x + 1$

51. $y_1 = \frac{1}{5}[10(x^2 - 1)]$

52. $y_1 = (x - 3) \cdot \frac{1}{x - 3}$

$y_2 = 2(x^2 - 1)$

$y_2 = 1$

Using a Graphing Utility to Graph an Equation In Exercises 53–56, use a graphing utility to graph the equation. Use the *trace* feature of the graphing utility to approximate the unknown coordinate of each solution point accurate to two decimal places. (Hint: You may need to use the *zoom* feature of the graphing utility to obtain the required accuracy.)

53. $y = \sqrt{5 - x}$

54. $y = x^3(x - 3)$

(a) $(3, y)$

(a) $(2.25, y)$

(b) $(x, 3)$

(b) $(x, 20)$

55. $y = x^5 - 5x$

56. $y = |x^2 - 6x + 5|$

(a) $(-0.5, y)$

(a) $(2, y)$

(b) $(x, -4)$

(b) $(x, 1.5)$

Using a Graphing Utility to Graph a Circle In Exercises 57–60, solve for y and use a graphing utility to graph each of the resulting equations in the same viewing window. (Adjust the viewing window so that the circle appears circular.)

✓ 57. $x^2 + y^2 = 16$

58. $x^2 + y^2 = 36$

59. $(x - 1)^2 + (y - 2)^2 = 9$

60. $(x - 3)^2 + (y - 1)^2 = 25$

Determining Solution Points In Exercises 61 and 62, determine which point lies on the graph of the circle. (There may be more than one correct answer.)

61. $(x - 1)^2 + (y - 2)^2 = 25$

(a) $(1, 3)$

(b) $(-2, 6)$

(c) $(5, -1)$

(d) $(0, 2 + 2\sqrt{6})$

62. $(x + 2)^2 + (y - 3)^2 = 25$

(a) $(-2, 3)$

(b) $(0, 0)$

(c) $(1, -1)$

(d) $(-1, 3 - 2\sqrt{6})$

✓ 63. MODELING DATA

A manufacturing plant purchases a new molding machine for \$225,000. The depreciated value (decreased value) y after t years is $y = 225,000 - 20,000t$, for $0 \leq t \leq 6$.

- Use the constraints of the model and a graphing utility to graph the equation using an appropriate viewing window.
- Use the *value* feature or the *zoom* and *trace* features of the graphing utility to determine the value of y when $t = 5.8$. Verify your answer algebraically.
- Use the *value* feature or the *zoom* and *trace* features of the graphing utility to determine the value of y when $t = 2.35$. Verify your answer algebraically.

64. MODELING DATA

You buy a personal watercraft for \$8100. The depreciated value y after t years is $y = 8100 - 929t$, for $0 \leq t \leq 6$.

- Use the constraints of the model and a graphing utility to graph the equation using an appropriate viewing window.
- Use the *zoom* and *trace* features of the graphing utility to determine the value of t when $y = 5545.25$. Verify your answer algebraically.
- Use the *value* feature or the *zoom* and *trace* features of the graphing utility to determine the value of y when $t = 5.5$. Verify your answer algebraically.

✓ 65. MODELING DATA

The table shows the median (middle) sales prices y (in thousands of dollars) of new one-family homes in the southern United States from 2000 through 2008. (Sources: U.S. Census Bureau; U.S. Department of Housing and Urban Development)



Year	Median sales price, y
2000	148.0
2001	155.4
2002	163.4
2003	168.1
2004	181.1
2005	197.3
2006	208.2
2007	217.7
2008	203.7

A model that represents the data is given by $y = -0.4221t^3 + 4.690t^2 - 3.47t + 150.9$, $0 \leq t \leq 8$, where t represents the year, with $t = 0$ corresponding to 2000.

- Use the model and the *table* feature of a graphing utility to find the median sales prices from 2000 through 2008. How well does the model fit the data? Explain.
- Use the graphing utility to graph the data from the table and the model in the same viewing window. How well does the model fit the data? Explain.
- Use the model to estimate the median sales prices in 2012 and 2014. Do the values seem reasonable? Explain.
- Use the *zoom* and *trace* features of the graphing utility to determine during which year(s) the median sales price was approximately \$150,000.

66. **Why you should learn it** (p. C10) The table shows the life expectancy y of a child (at birth) in the United States for each of the selected years from 1930 through 2000. (Source: U.S. National Center for Health Statistics)

Year	Life expectancy, y
1930	59.7
1940	62.9
1950	68.2
1960	69.7
1970	70.8
1980	73.7
1990	75.4
2000	77.0

A model that represents the data is given by

$$y = \frac{59.617 + 1.18t}{1 + 0.012t}, \quad 0 \leq t \leq 70$$

where t is the time in years, with $t = 0$ corresponding to 1930.

- Use a graphing utility to graph the data from the table above and the model in the same viewing window. How well does the model fit the data? Explain.
- Find the y -intercept of the graph of the model. What does it represent in the context of the problem?
- Use the *zoom* and *trace* features of the graphing utility to determine the year when the life expectancy was 73.2. Verify your answer algebraically.
- Determine the life expectancy in 1948 both graphically and algebraically.
- Use the model to estimate the life expectancy of a child born in 2020.

Conclusions

True or False? In Exercises 67 and 68, determine whether the statement is true or false. Justify your answer.

- A parabola can have only one x -intercept.
- The graph of a linear equation can have either no x -intercepts or only one x -intercept.
- Writing** Your employer offers you a choice of wage scales: a monthly salary of \$3000 plus commission of 7% of sales or a salary of \$3400 plus a 5% commission. Write a short paragraph discussing how you would choose your option. At what sales level would the options yield the same salary?
- Finance** You open a savings account and deposit \$200. Every week you withdraw \$50. The account balance y after t weeks is $y = -50t + 200$, for $0 \leq t \leq 4$.
 - Use point plotting and graph paper to sketch the graph of $y = -50t + 200$.
 - Use a graphing utility to graph $y = -50t + 200$.
 - Explain how to find an appropriate viewing window for the graph of the equation.
 - Find the y -intercept of the graph of the model. What does it represent in the context of the problem?

C.3 Solving Equations Algebraically and Graphically

Equations and Solutions of Equations

An **equation** in x is a statement that two algebraic expressions are equal. For example,

$$3x - 5 = 7, \quad x^2 - x - 6 = 0, \quad \text{and} \quad \sqrt{2x} = 4$$

are equations. To **solve** an equation in x means to find all values of x for which the equation is true. Such values are **solutions**. For instance, $x = 4$ is a solution of the equation $3x - 5 = 7$, because $3(4) - 5 = 7$ is a true statement.

The solutions of an equation depend on the kinds of numbers being considered. For instance, in the set of rational numbers, $x^2 = 10$ has no solution because there is no rational number whose square is 10. In the set of real numbers, however, the equation has two solutions: $x = \sqrt{10}$ and $x = -\sqrt{10}$.

An equation can be classified as an *identity*, a *conditional*, or a *contradiction*, as shown in the following table.

Equation	Definition	Example
Identity	An equation that is true for every real number in the domain of the variable	$\frac{x}{3x^2} = \frac{1}{3x}$ is a true statement for any nonzero real value of x .
Conditional	An equation that is true for just some (but not all) of the real numbers in the domain of the variable	$x^2 - 9 = 0$ is a true statement for $x = 3$ and $x = -3$, but not for any other real values.
Contradiction	An equation that is false for every real number in the domain of the variable	$2x + 1 = 2x - 3$ is a false statement for any real value of x .

A **linear equation in one variable** x is an equation that can be written in the standard form $ax + b = 0$, where a and b are real numbers, with $a \neq 0$.

To solve an equation involving fractional expressions, find the least common denominator (LCD) of all terms in the equation and multiply every term by this LCD. This procedure clears the equation of fractions.

Example 1 Solving an Equation Involving Fractions

Solve $\frac{x}{3} + \frac{3x}{4} = 2$.

Solution

$$\frac{x}{3} + \frac{3x}{4} = 2$$

Write original equation.

$$(12) \frac{x}{3} + (12) \frac{3x}{4} = (12)2$$

Multiply each term by the LCD of 12.

$$4x + 9x = 24$$

Divide out and multiply.

$$13x = 24$$

Combine like terms.

$$x = \frac{24}{13}$$

Divide each side by 13.

 **CHECKPOINT** Now try Exercise 23.

What you should learn

- Solve linear equations.
- Find x - and y -intercepts of graphs of equations.
- Find solutions of equations graphically.
- Find the points of intersection of two graphs.
- Solve polynomial equations.
- Solve equations involving radicals, fractions, or absolute values.

Why you should learn it

Knowing how to solve equations algebraically and graphically can help you solve real-life problems. For instance, in Exercise 195 on page C34, you will find the point of intersection of the graphs of two population models both algebraically and graphically.

Study Tip



For a review of solving one- and two-step linear equations, see Appendix E at this textbook's Companion Website.

After solving an equation, you should check each solution in the original equation. For instance, you can check the solution to Example 1 as follows.

$$\frac{x}{3} + \frac{3x}{4} = 2 \quad \text{Write original equation.}$$

$$\frac{\frac{24}{13}}{3} + \frac{3(\frac{24}{13})}{4} \stackrel{?}{=} 2 \quad \text{Substitute } \frac{24}{13} \text{ for } x.$$

$$\frac{8}{13} + \frac{18}{13} \stackrel{?}{=} 2 \quad \text{Simplify.}$$

$$2 = 2 \quad \text{Solution checks. } \checkmark$$

When multiplying or dividing an equation by a *variable* expression, it is possible to introduce an **extraneous solution**—one that does not satisfy the original equation. The next example demonstrates the importance of checking your solution when you have multiplied or divided by a variable expression.

Example 2 An Equation with an Extraneous Solution

Solve $\frac{1}{x-2} = \frac{3}{x+2} - \frac{6x}{x^2-4}$.

Solution

The LCD is $x^2 - 4 = (x + 2)(x - 2)$. Multiplying each term by the LCD and simplifying produces the following.

$$\frac{1}{x-2}(x+2)(x-2) = \frac{3}{x+2}(x+2)(x-2) - \frac{6x}{x^2-4}(x+2)(x-2)$$

$$x+2 = 3(x-2) - 6x, \quad x \neq \pm 2$$

$$x+2 = 3x-6-6x$$

$$4x = -8$$

$$x = -2 \quad \text{Extraneous solution}$$

In the original equation, $x = -2$ yields a denominator of zero. So, $x = -2$ is an extraneous solution, and the original equation has *no solution*.

 **CHECKPOINT** Now try Exercise 39.

Study Tip



Recall that the least common denominator (LCD) of several

rational expressions consists of the product of all prime factors in the denominators, with each factor given the highest power of its occurrence in any denominator.

Intercepts and Solutions

In Appendix C.2, you learned that the intercepts of a graph are the points at which the graph intersects the x - or y -axis.

Definitions of Intercepts

1. The point $(a, 0)$ is called an **x -intercept** of the graph of an equation when it is a solution point of the equation. To find the x -intercept(s), set y equal to 0 and solve the equation for x .
2. The point $(0, b)$ is called a **y -intercept** of the graph of an equation when it is a solution point of the equation. To find the y -intercept(s), set x equal to 0 and solve the equation for y .

Sometimes it is convenient to denote the x -intercept as simply the x -coordinate of the point $(a, 0)$ rather than the point itself. Unless it is necessary to make a distinction, “intercept” will be used to mean either the point or the coordinate.

It is possible for a graph to have no intercepts, one intercept, or several intercepts. For instance, consider the four graphs shown in Figure C.26.

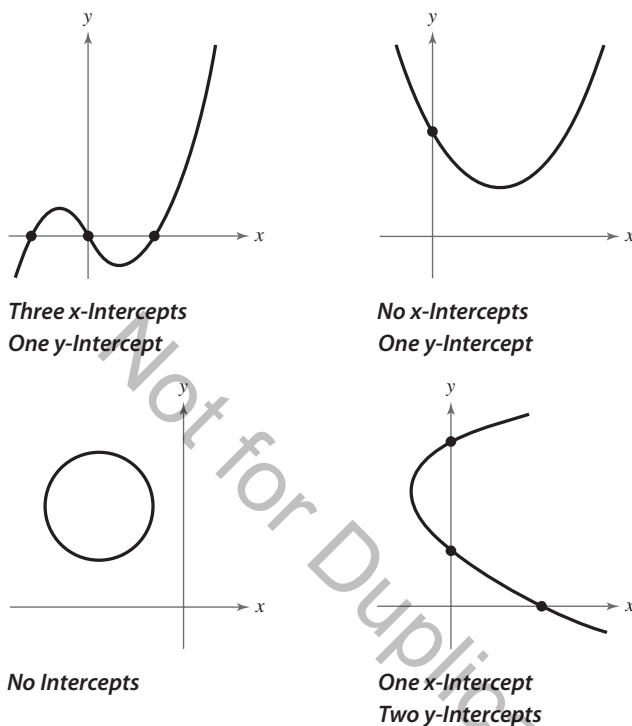


Figure C.26

Example 3 Finding x - and y -Intercepts

Find the x - and y -intercepts of the graph of $2x + 3y = 5$.

Solution

To find the x -intercept, let $y = 0$ and solve for x . This produces

$$2x = 5 \quad \Rightarrow \quad x = \frac{5}{2}$$

which implies that the graph has one x -intercept at $(\frac{5}{2}, 0)$. To find the y -intercept, let $x = 0$ and solve for y . This produces

$$3y = 5 \quad \Rightarrow \quad y = \frac{5}{3}$$

which implies that the graph has one y -intercept at $(0, \frac{5}{3})$. See Figure C.27.

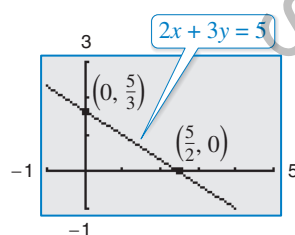


Figure C.27

CHECKPOINT Now try Exercise 41.

The concepts of x -intercepts and solutions of equations are closely related. In fact, the following statements are equivalent.

1. The point $(a, 0)$ is an x -intercept of the graph of $y = f(x)$.
2. The number a is a solution of the equation $f(x) = 0$.

The close connection among x -intercepts and solutions is crucial to your study of algebra. You can take advantage of this connection in two ways. Use your algebraic “equation-solving skills” to find the x -intercepts of a graph and your “graphing skills” to approximate the solutions of an equation.

Finding Solutions Graphically

Polynomial equations of degree 1 or 2 can be solved in relatively straightforward ways. Solving polynomial equations of higher degrees can, however, be quite difficult, especially when you rely only on algebraic techniques. For such equations, a graphing utility can be very helpful.

Graphical Approximations of Solutions of an Equation

1. Write the equation in *general form*, $f(x) = 0$, with the nonzero terms on one side of the equation and zero on the other side.
2. Use a graphing utility to graph the function $y = f(x)$. Be sure the viewing window shows all the relevant features of the graph.
3. Use the *zero* or *root* feature or the *zoom* and *trace* features of the graphing utility to approximate the x -intercepts of the graph of f .

Chapter 2 shows techniques for determining the number of solutions of a polynomial equation. For now, you should know that a polynomial equation of degree n cannot have more than n different solutions.

Example 4 Finding Solutions of an Equation Graphically

Use a graphing utility to approximate the solutions of

$$2x^3 - 3x + 2 = 0.$$

Solution

Graph the function $y = 2x^3 - 3x + 2$. You can see from the graph that there is one x -intercept. It lies between -2 and -1 and is approximately -1.5 . By using the *zero* or *root* feature of a graphing utility, you can improve the approximation. Choose a left bound of $x = -2$ (see Figure C.28) and a right bound of $x = -1$ (see Figure C.29). To two-decimal-place accuracy, the solution is $x \approx -1.48$, as shown in Figure C.30. Check this approximation on your calculator. You will find that the value of y is

$$y = 2(-1.48)^3 - 3(-1.48) + 2 \approx -0.04.$$

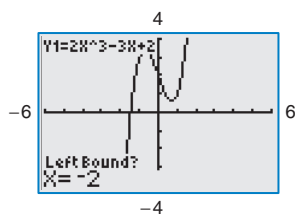


Figure C.28

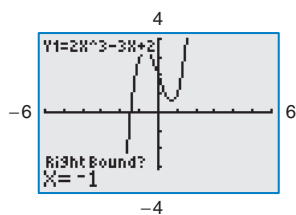


Figure C.29

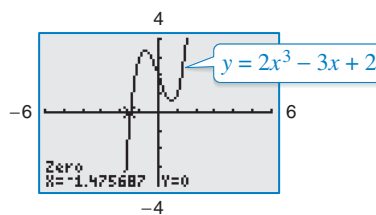


Figure C.30

CHECKPOINT Now try Exercise 69.

You can also use a graphing utility's *zoom* and *trace* features to approximate the solution of an equation. Here are some suggestions for using the *zoom-in* feature of a graphing utility.

1. With each successive zoom-in, adjust the x -scale (if necessary) so that the resulting viewing window shows at least one scale mark on each side of the solution.
2. The accuracy of the approximation will always be such that the error is less than the distance between two scale marks. For instance, to approximate the zero to the nearest hundredth, set the x -scale to 0.01. To approximate the zero to the nearest thousandth, set the x -scale to 0.001.
3. The graphing utility's *trace* feature can sometimes be used to add one more decimal place of accuracy without changing the viewing window.

Unless stated otherwise, all real solutions in this text will be approximated with an error of *at most* 0.01.

Example 5 Approximating Solutions of an Equation Graphically

Use a graphing utility to approximate the solutions of

$$x^2 + 3 = 5x.$$

Solution

In general form, this equation is

$$x^2 - 5x + 3 = 0. \quad \text{Equation in general form}$$

So, you can begin by graphing

$$y = x^2 - 5x + 3 \quad \text{Function to be graphed}$$

as shown in Figure C.31. This graph has two x -intercepts, and by using the *zoom* and *trace* features you can approximate the corresponding solutions to be

$$x \approx 0.70 \quad \text{and} \quad x \approx 4.30$$

as shown in Figures C.32 and C.33.

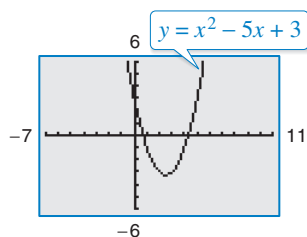


Figure C.31

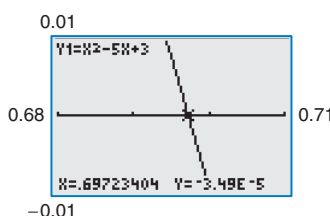


Figure C.32

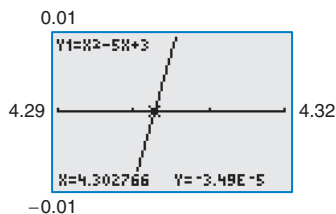


Figure C.33

 **CHECKPOINT** Now try Exercise 79.

Technology Tip



Use the *zero* or *root* feature of a graphing utility to approximate the solutions of the equation in Example 5 to see that it yields a similar result.

Points of Intersection of Two Graphs

An ordered pair that is a solution of two different equations is called a **point of intersection** of the graphs of the two equations. For instance, in Figure C.34 you can see that the graphs of the following equations have two points of intersection.

$$y = x + 2 \quad \text{Equation 1}$$

$$y = x^2 - 2x - 2 \quad \text{Equation 2}$$

The point $(-1, 1)$ is a solution of both equations, and the point $(4, 6)$ is a solution of both equations. To check this algebraically, substitute

$$x = -1 \quad \text{and} \quad x = 4$$

into each equation.

Check that $(-1, 1)$ is a solution.

$$\text{Equation 1: } y = -1 + 2 = 1 \quad \text{Solution checks. } \checkmark$$

$$\text{Equation 2: } y = (-1)^2 - 2(-1) - 2 = 1 \quad \text{Solution checks. } \checkmark$$

Check that $(4, 6)$ is a solution.

$$\text{Equation 1: } y = 4 + 2 = 6 \quad \text{Solution checks. } \checkmark$$

$$\text{Equation 2: } y = (4)^2 - 2(4) - 2 = 6 \quad \text{Solution checks. } \checkmark$$

To find the points of intersection of the graphs of two equations, solve each equation for y (or x) and set the two results equal to each other. The resulting equation will be an equation in one variable, which can be solved using standard procedures, as shown in Example 6.

Example 6 Finding Points of Intersection

Find any points of intersection of the graphs of

$$2x - 3y = -2 \quad \text{and} \quad 4x - y = 6.$$

Algebraic Solution

To begin, solve each equation for y to obtain

$$y = \frac{2}{3}x + \frac{2}{3} \quad \text{and} \quad y = 4x - 6.$$

Next, set the two expressions for y equal to each other and solve the resulting equation for x , as follows.

$$\frac{2}{3}x + \frac{2}{3} = 4x - 6 \quad \text{Equate expressions for } y.$$

$$2x + 2 = 12x - 18 \quad \text{Multiply each side by 3.}$$

$$-10x = -20 \quad \text{Subtract } 12x \text{ and } 2 \text{ from each side.}$$

$$x = 2 \quad \text{Divide each side by } -10.$$

When $x = 2$, the y -value of each of the original equations is 2. So, the point of intersection is $(2, 2)$.

 **CHECKPOINT** Now try Exercise 87.

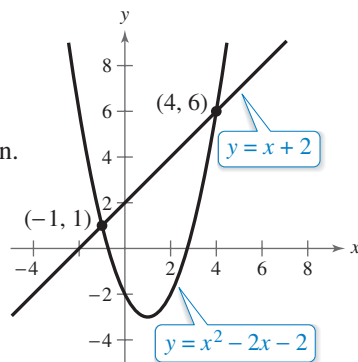


Figure C.34

Graphical Solution

To begin, solve each equation for y to obtain

$$y_1 = \frac{2}{3}x + \frac{2}{3} \quad \text{and} \quad y_2 = 4x - 6.$$

Then use a graphing utility to graph both equations in the same viewing window, as shown in Figure C.35. Use the *intersect* feature to approximate the point of intersection.

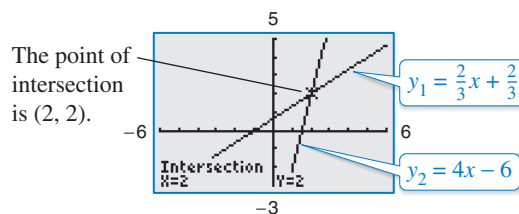


Figure C.35

Another way to approximate the points of intersection of two graphs is to graph both equations with a graphing utility and use the *zoom* and *trace* features to find the point or points at which the two graphs intersect.

Example 7 Approximating Points of Intersection Graphically

Use a graphing utility to approximate any points of intersection of the graphs of the following equations.

$$y = x^2 - 3x - 4 \quad \text{Equation 1 (quadratic function)}$$

$$y = x^3 + 3x^2 - 2x - 1 \quad \text{Equation 2 (cubic function)}$$

Solution

Begin by using the graphing utility to graph both functions, as shown in Figure C.36. From this display, you can see that the two graphs have only one point of intersection. Then, using the *zoom* and *trace* features, approximate the point of intersection to be

$$(-2.17, 7.25) \quad \text{Point of intersection}$$

as shown in Figure C.37.

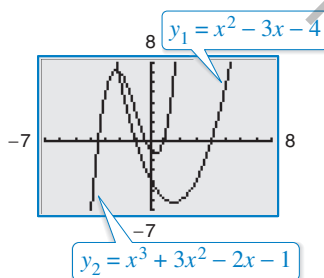


Figure C.36

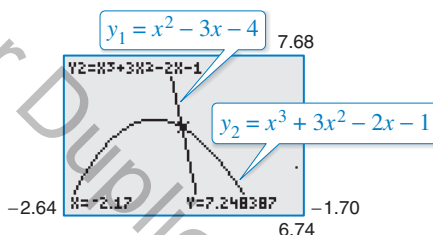


Figure C.37

To test the reasonableness of this approximation, you can evaluate both functions at $x = -2.17$.

Quadratic Function

$$\begin{aligned} y &= (-2.17)^2 - 3(-2.17) - 4 \\ &\approx 7.22 \end{aligned}$$

Cubic Function

$$\begin{aligned} y &= (-2.17)^3 + 3(-2.17)^2 - 2(-2.17) - 1 \\ &\approx 7.25 \end{aligned}$$

Because both functions yield approximately the same y -value, you can conclude that the approximate coordinates of the point of intersection are

$$x \approx -2.17 \quad \text{and} \quad y \approx 7.25.$$

CHECKPOINT Now try Exercise 97.

Technology Tip



The table shows some points on the graphs of the equations in

Example 6. You can find any points of intersection of the graphs by finding all values of x for which y_1 and y_2 are equal.

X	Y ₁	Y ₂
-1	0	-10
0	.66667	.6
1	1.33333	2
2	2	5
3	2.66667	10
4	3.33333	14
X=2		

Technology Tip



Use the *intersect* feature of a graphing utility to approximate the point of intersection in Example 7 to see that it yields a similar result.

Solving Polynomial Equations Algebraically

Polynomial equations can be classified by their *degree*. The degree of a polynomial equation is the highest degree of its terms.

<i>Degree</i>	<i>Name</i>	<i>Example</i>
First	Linear equation	$6x + 2 = 4$
Second	Quadratic equation	$2x^2 - 5x + 3 = 0$
Third	Cubic equation	$x^3 - x = 0$
Fourth	Quartic equation	$x^4 - 3x^2 + 2 = 0$
Fifth	Quintic equation	$x^5 - 12x^2 + 7x + 4 = 0$

You should be familiar with the following four methods for solving quadratic equations *algebraically*.

Solving a Quadratic Equation

Factoring: If $ab = 0$, then $a = 0$ or $b = 0$.

Zero-Factor Property

Example: $x^2 - x - 6 = 0$

$$(x - 3)(x + 2) = 0$$

$$x - 3 = 0 \quad \Rightarrow \quad x = 3$$

$$x + 2 = 0 \quad \Rightarrow \quad x = -2$$

Extracting Square Roots: If $u^2 = c$, then $u = \pm\sqrt{c}$.

Example: $(x + 3)^2 = 16$

$$x + 3 = \pm 4$$

$$x = -3 \pm 4$$

$$x = 1 \quad \text{or} \quad x = -7$$

Completing the Square: If $x^2 + bx = c$, then

$$x^2 + bx + \left(\frac{b}{2}\right)^2 = c + \left(\frac{b}{2}\right)^2$$

$$\left(x + \frac{b}{2}\right)^2 = c + \frac{b^2}{4}$$

Example: $x^2 + 6x = 5$

$$x^2 + 6x + 3^2 = 5 + 3^2$$

$$(x + 3)^2 = 14$$

$$x + 3 = \pm\sqrt{14}$$

$$x = -3 \pm \sqrt{14}$$

Quadratic Formula: If $ax^2 + bx + c = 0$, then $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$.

Example: $2x^2 + 3x - 1 = 0$

$$x = \frac{-3 \pm \sqrt{3^2 - 4(2)(-1)}}{2(2)}$$

$$x = \frac{-3 \pm \sqrt{17}}{4}$$

The methods used to solve quadratic equations can sometimes be extended to polynomial equations of higher degree, as shown in the next two examples.

Example 8 Solving a Polynomial Equation by FactoringSolve $2x^3 - 6x^2 - 6x + 18 = 0$.**Solution**

This equation has a common factor of 2. You can simplify the equation by first dividing each side of the equation by 2.

$$2x^3 - 6x^2 - 6x + 18 = 0$$

Write original equation.

$$x^3 - 3x^2 - 3x + 9 = 0$$

Divide each side by 2.

$$x^2(x - 3) - 3(x - 3) = 0$$

Group terms.

$$(x - 3)(x^2 - 3) = 0$$

Factor by grouping.

$$x - 3 = 0 \quad \Rightarrow \quad x = 3$$

Set 1st factor equal to 0.

$$x^2 - 3 = 0 \quad \Rightarrow \quad x = \pm\sqrt{3}$$

Set 2nd factor equal to 0.

The equation has three solutions: $x = 3$, $x = \sqrt{3}$, and $x = -\sqrt{3}$. Check these solutions in the original equation. Figure C.38 verifies the solutions graphically.

CHECKPOINT Now try Exercise 155.

Study Tip

Many cubic polynomial equations can be solved using factoring by grouping, as illustrated in Example 8.

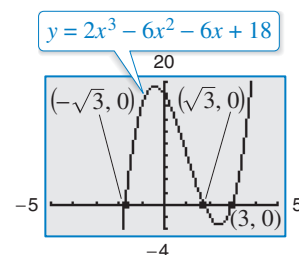


Figure C.38

Occasionally, mathematical models involve equations that are of **quadratic type**. In general, an equation is of quadratic type when it can be written in the form

$$au^2 + bu + c = 0$$

where $a \neq 0$ and u is an algebraic expression.

Example 9 Solving an Equation of Quadratic TypeSolve $x^4 - 3x^2 + 2 = 0$.**Solution**

This equation is of quadratic type with $u = x^2$.

$$(x^2)^2 - 3(x^2) + 2 = 0$$

To solve this equation, you can use the Quadratic Formula.

$$x^4 - 3x^2 + 2 = 0$$

Write original equation.

$$\underbrace{(x^2)^2}_{u^2} - \underbrace{3(x^2)}_{3u} + 2 = 0$$

Write in quadratic form.

$$x^2 = \frac{-(-3) \pm \sqrt{(-3)^2 - 4(1)(2)}}{2(1)}$$

Quadratic Formula

$$x^2 = \frac{3 \pm 1}{2}$$

Simplify.

$$x^2 = 2 \quad \Rightarrow \quad x = \pm\sqrt{2}$$

Solutions

$$x^2 = 1 \quad \Rightarrow \quad x = \pm 1$$

Solutions

The equation has four solutions: $x = -1$, $x = 1$, $x = \sqrt{2}$, and $x = -\sqrt{2}$. Check these solutions in the original equation. Figure C.39 verifies the solutions graphically.

CHECKPOINT Now try Exercise 157.

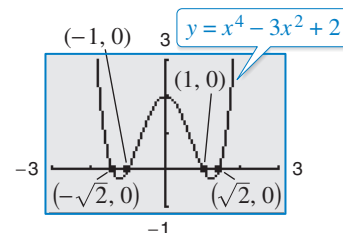


Figure C.39

Equations Involving Radicals

An equation involving a radical expression can often be cleared of radicals by raising each side of the equation to an appropriate power. When using this procedure, remember to check for extraneous solutions.

Example 10 Solving Equations Involving Radicals

a. $\sqrt{2x + 7} - x = 2$

$$\sqrt{2x + 7} = x + 2$$

$$2x + 7 = x^2 + 4x + 4$$

$$x^2 + 2x - 3 = 0$$

$$(x + 3)(x - 1) = 0$$

$$x + 3 = 0 \quad \Rightarrow \quad x = -3$$

$$x - 1 = 0 \quad \Rightarrow \quad x = 1$$

By checking these values, you can determine that the only solution is $x = 1$. Figure C.40 verifies the solution graphically.

Original equation

Isolate radical.

Square each side.

Write in general form.

Factor.

Set 1st factor equal to 0.

Set 2nd factor equal to 0.

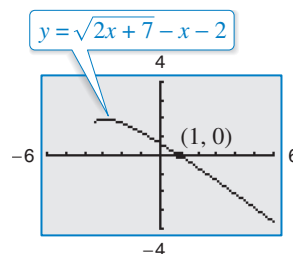


Figure C.40

b. $\sqrt{2x + 6} - \sqrt{x + 4} = 1$

$$\sqrt{2x + 6} = 1 + \sqrt{x + 4}$$

$$2x + 6 = 1 + 2\sqrt{x + 4} + (x + 4)$$

$$x + 1 = 2\sqrt{x + 4}$$

$$x^2 + 2x + 1 = 4(x + 4)$$

$$x^2 - 2x - 15 = 0$$

$$(x - 5)(x + 3) = 0$$

$$x - 5 = 0 \quad \Rightarrow \quad x = 5$$

$$x + 3 = 0 \quad \Rightarrow \quad x = -3$$

By checking these values, you can determine that the only solution is $x = 5$. Figure C.41 verifies the solution graphically.

Original equation

Isolate $\sqrt{2x + 6}$.

Square each side.

Isolate $2\sqrt{x + 4}$.

Square each side.

Write in general form.

Factor.

Set 1st factor equal to 0.

Set 2nd factor equal to 0.

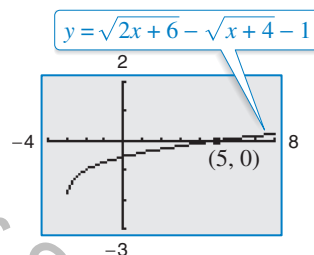


Figure C.41

CHECKPOINT Now try Exercise 169.

Example 11 Solving an Equation Involving a Rational Exponent

Solve $(x + 1)^{2/3} = 4$.

Solution

$$(x + 1)^{2/3} = 4$$

$$\sqrt[3]{(x + 1)^2} = 4$$

$$(x + 1)^2 = 64$$

$$x + 1 = \pm 8$$

$$x = 7, x = -9$$

The solutions are $x = 7$ and $x = -9$. Check these in the original equation. Figure C.42 verifies the solutions graphically.

Write original equation.

Rewrite in radical form.

Cube each side.

Take square root of each side.

Subtract 1 from each side.

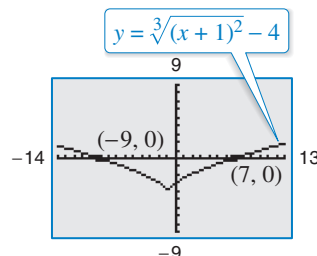


Figure C.42

CHECKPOINT Now try Exercise 175.

Equations Involving Fractions or Absolute Values

As demonstrated in Example 1, you can solve an equation involving fractions algebraically by multiplying each side of the equation by the least common denominator of all terms in the equation to clear the equation of fractions.

Example 12 Solving an Equation Involving Fractions

Solve $\frac{2}{x} = \frac{3}{x-2} - 1$.

Solution

For this equation, the least common denominator of the three terms is

$$x(x-2)$$

so you can begin by multiplying each term of the equation by this expression.

$$\frac{2}{x} = \frac{3}{x-2} - 1 \quad \text{Write original equation.}$$

$$x(x-2)\frac{2}{x} = x(x-2)\frac{3}{x-2} - x(x-2)(1) \quad \text{Multiply each term by the LCD.}$$

$$2(x-2) = 3x - x(x-2), \quad x \neq 0, 2 \quad \text{Simplify.}$$

$$2x - 4 = -x^2 + 5x \quad \text{Simplify.}$$

$$x^2 - 3x - 4 = 0 \quad \text{Write in general form.}$$

$$(x-4)(x+1) = 0 \quad \text{Factor.}$$

$$x-4 = 0 \quad \Rightarrow \quad x = 4 \quad \text{Set 1st factor equal to 0.}$$

$$x+1 = 0 \quad \Rightarrow \quad x = -1 \quad \text{Set 2nd factor equal to 0.}$$

Check these solutions in the original equation as follows.

Check $x = 4$

$$\frac{2}{x} = \frac{3}{x-2} - 1$$

$$\frac{2}{4} \stackrel{?}{=} \frac{3}{4-2} - 1$$

$$\frac{1}{2} \stackrel{?}{=} \frac{3}{2} - 1$$

$$\frac{1}{2} = \frac{1}{2} \quad \checkmark$$

Check $x = -1$

$$\frac{2}{x} = \frac{3}{x-2} - 1$$

$$\frac{2}{-1} \stackrel{?}{=} \frac{3}{-1-2} - 1$$

$$-2 \stackrel{?}{=} -1 - 1$$

$$-2 = -2 \quad \checkmark$$

So, the solutions are $x = 4$ and $x = -1$.

 **CHECKPOINT** Now try Exercise 179.

Example 13 Solving an Equation Involving an Absolute ValueSolve $|x^2 - 3x| = -4x + 6$.**Solution**

To solve an equation involving an absolute value algebraically, you must consider the fact that the expression inside the absolute value symbols can be positive or negative. This results in *two* separate equations, each of which must be solved.

First Equation

$$x^2 - 3x = -4x + 6$$

Use positive expression.

$$x^2 + x - 6 = 0$$

Write in general form.

$$(x + 3)(x - 2) = 0$$

Factor.

$$x + 3 = 0 \quad \Rightarrow \quad x = -3$$

Set 1st factor equal to 0.

$$x - 2 = 0 \quad \Rightarrow \quad x = 2$$

Set 2nd factor equal to 0.

Second Equation

$$-(x^2 - 3x) = -4x + 6$$

Use negative expression.

$$x^2 - 7x + 6 = 0$$

Write in general form.

$$(x - 1)(x - 6) = 0$$

Factor.

$$x - 1 = 0 \quad \Rightarrow \quad x = 1$$

Set 1st factor equal to 0.

$$x - 6 = 0 \quad \Rightarrow \quad x = 6$$

Set 2nd factor equal to 0.

Check

$$|(-3)^2 - 3(-3)| \stackrel{?}{=} -4(-3) + 6$$

Substitute -3 for x .

$$18 = 18$$

 -3 checks. ✓

$$|2^2 - 3(2)| \stackrel{?}{=} -4(2) + 6$$

Substitute 2 for x .

$$2 \neq -2$$

 2 does not check.

$$|1^2 - 3(1)| \stackrel{?}{=} -4(1) + 6$$

Substitute 1 for x .

$$2 = 2$$

 1 checks. ✓

$$|6^2 - 3(6)| \stackrel{?}{=} -4(6) + 6$$

Substitute 6 for x .

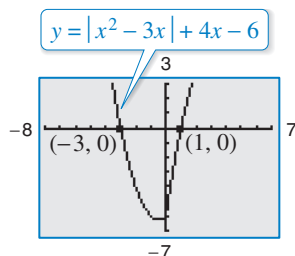
$$18 \neq -18$$

 6 does not check.

The solutions are

$$x = -3 \quad \text{and} \quad x = 1.$$

Figure C.43 shows the solutions graphically.

**Figure C.43**

C.3 Exercises

For instructions on how to use a graphing utility, see Appendix A.

Vocabulary and Concept Check

In Exercises 1–4, fill in the blank.

1. An _____ is a statement that two algebraic expressions are equal.
2. To find all values for which an equation is true is to _____ the equation.
3. When solving an equation, it is possible to introduce an _____ solution, which is a value that does not satisfy the original equation.
4. An ordered pair that is a solution of two different equations represents a _____ of the graphs of the two equations.

Procedures and Problem Solving

Checking Solutions In Exercises 5–10, determine whether each value of x is a solution of the equation.

Equation	Values
5. $\frac{5}{2x} - \frac{4}{x} = 3$	(a) $x = -\frac{1}{2}$ (b) $x = 4$ (c) $x = 0$ (d) $x = \frac{1}{4}$
6. $\frac{x}{2} + \frac{6x}{7} = \frac{19}{14}$	(a) $x = -2$ (b) $x = 1$ (c) $x = \frac{1}{2}$ (d) $x = 7$
7. $3 + \frac{1}{x+2} = 4$	(a) $x = -1$ (b) $x = -2$ ✓ (c) $x = 0$ (d) $x = 5$
8. $\frac{(x+5)(x-3)}{2} = 24$	(a) $x = -3$ (b) $x = -2$ (c) $x = 7$ (d) $x = 9$
9. $\frac{\sqrt{x+4}}{6} + 3 = 4$	(a) $x = -3$ (b) $x = 0$ (c) $x = 21$ (d) $x = 32$
10. $\frac{\sqrt[3]{x-8}}{3} = -\frac{2}{3}$	(a) $x = -16$ (b) $x = 0$ (c) $x = 9$ (d) $x = 16$

Classifying an Equation In Exercises 11–16, determine whether the equation is an identity or a conditional equation.

11. $2(x-1) = 2x-2$
12. $-7(x-3) + 4x = 3(7-x)$
13. $x^2 - 8x + 5 = (x-4)^2 - 11$
14. $x^2 + 2(3x-2) = x^2 + 6x - 4$
15. $3 + \frac{1}{x+1} = \frac{4x}{x+1}$
16. $\frac{5}{x} + \frac{3}{x} = 24$

Solving an Equation In Exercises 17–40, solve the equation (if possible). Use a graphing utility to verify your solution.

17. $3x - 5 = 2x + 7$
18. $5x + 3 = 6 - 2x$
19. $4y + 2 - 5y = 7 - 6y$
20. $5y + 1 = 8y - 5 + 6y$
21. $3(y-5) = 3 + 5y$
22. $5(z-4) + 4z = 5 - 6z$
23. $\frac{x}{5} - \frac{x}{2} = 3$
24. $\frac{5x}{4} + \frac{1}{2} = x - \frac{1}{2}$
25. $\frac{3}{2}(z+5) - \frac{1}{4}(z+24) = 0$
26. $\frac{3x}{2} + \frac{1}{4}(x-2) = 10$
27. $\frac{2(z-4)}{5} + 5 = 10z$
28. $\frac{5}{3} + 2(y+1) = \frac{10}{3}$
29. $\frac{100-4u}{3} = \frac{5u+6}{4} + 6$
30. $\frac{17+y}{y} + \frac{32+y}{y} = 100$
31. $\frac{5x-4}{5x+4} = \frac{2}{3}$
32. $\frac{10x+3}{5x+6} = \frac{1}{2}$
33. $\frac{1}{x-3} + \frac{1}{x+3} = \frac{10}{x^2-9}$
34. $\frac{1}{x-2} + \frac{3}{x+3} = \frac{4}{x^2+x-6}$

35. $\frac{7}{2x+1} - \frac{8x}{2x-1} = -4$

36. $\frac{x}{x+4} + \frac{4}{x+4} + 2 = 0$

37. $\frac{1}{x} + \frac{2}{x-5} = 0$

38. $3 = 2 + \frac{2}{z+2}$

✓ 39. $\frac{3}{x^2-3x} + \frac{4}{x} = \frac{1}{x-3}$

40. $\frac{6}{x} - \frac{2}{x+3} = \frac{3(x+5)}{x(x+3)}$

Finding x- and y-Intercepts In Exercises 41–52, find the x- and y-intercepts of the graph of the equation algebraically.

✓ 41. $y = x - 5$

42. $y = -\frac{3}{4}x - 3$

43. $y = x^2 + x - 2$

44. $y = 4 - x^2$

45. $y = x\sqrt{x+2}$

46. $y = -\frac{1}{2}x\sqrt{x+3}$

47. $y = \frac{4}{x}$

48. $y = \frac{3x-1}{4x}$

49. $y = |x-2| - 4$

50. $y = 3 - \frac{1}{2}|x+1|$

51. $xy - 2y - x + 1 = 0$

52. $xy - x + 4y = 0$

Comparing x-Intercepts and Solutions of an Equation

In Exercises 53–56, use a graphing utility to graph the equation and approximate any x-intercepts. Set $y = 0$ and solve the resulting equation. Compare the results with the x-intercepts of the graph.

53. $y = 2(x-1) - 4$

54. $y = 4(x+3) - 2$

55. $y = 20 - (3x-10)$

56. $y = 10 + 2(x-2)$

Checking Solutions In Exercises 57–62, the solution(s) of the equation are given. Verify the solution(s) both algebraically and graphically.

Equation

Solution(s)

57. $y = 5(4-x)$

$x = 4$

58. $y = 3(x-5) + 9$

$x = 2$

59. $y = x^3 - 6x^2 + 5x$

$x = 0, 5, 1$

60. $y = x^3 - 9x^2 + 18x$

$x = 0, 3, 6$

61. $y = \frac{x+2}{3} - \frac{x-1}{5} - 1$

$x = 1$

62. $y = x - 3 - \frac{10}{x}$

$x = -2, 5$

Finding Solutions of an Equation Graphically In Exercises 63–86, write the equation in the form $y = 0$ (if necessary) and use a graphing utility to approximate the solutions of the equation. Use a calculator to verify your answer.

63. $2.7x - 0.4x = 1.2$

64. $3.5x - 8 = 0.5x$

65. $25(x-3) = 12(x+2) - 10$

66. $1200 = 300 + 2(x-500)$

67. $\frac{3x}{2} + \frac{1}{4}(x-2) = 10$

68. $\frac{2x}{3} + \frac{1}{2}(x-5) = 6$

✓ 69. $x^3 - x + 4 = 0$

70. $x^3 - 9x^2 + 26x - 27 = 0$

71. $\frac{2x}{3} = 10 - \frac{24}{x}$

72. $\frac{x-3}{25} = \frac{x-5}{12}$

73. $\frac{3}{x+2} - \frac{4}{x-2} = 5$

74. $\frac{6}{x} + \frac{8}{x+5} = 3$

75. $(x+2)^2 = x^2 - 6x + 1$

76. $(x+1)^2 + 2(x-2) = (x+1)(x-2)$

77. $2x^3 - x^2 - 18x + 9 = 0$

78. $4x^3 + 12x^2 - 8x - 24 = 0$

✓ 79. $x^4 = 2x^2 - 1$

80. $5 = 3x^{1/3} + 2x^{2/3}$

81. $\frac{2}{x+2} = 3$

82. $\frac{5}{x} = 1 + \frac{3}{x+2}$

83. $|x-3| = 4$

84. $|x+1| = 6$

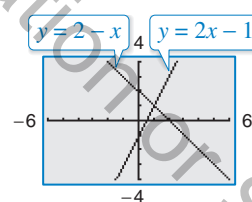
85. $\sqrt{x-2} = 3$

86. $\sqrt{x-4} = 8$

Finding Points of Intersection In Exercises 87–92, determine any point(s) of intersection of the equations algebraically. Then use a graphing utility to verify your results.

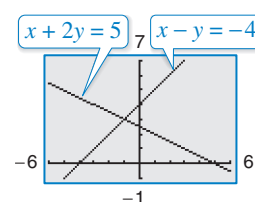
✓ 87. $y = 2 - x$

$y = 2x - 1$



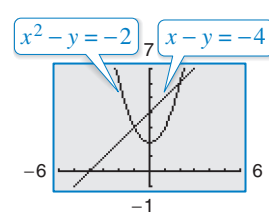
88. $x - y = -4$

$x + 2y = 5$



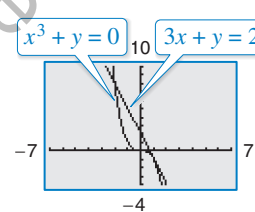
89. $x - y = -4$

$x^2 - y = -2$



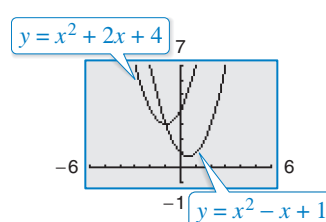
90. $3x + y = 2$

$x^3 + y = 0$



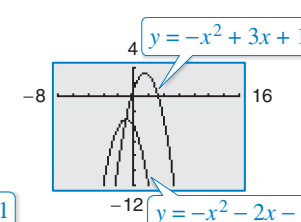
91. $y = x^2 - x + 1$

$y = x^2 + 2x + 4$



92. $y = -x^2 + 3x + 1$

$y = -x^2 - 2x - 4$



Approximating Points of Intersection Graphically In Exercises 93–98, use a graphing utility to approximate any points of intersection (accurate to three decimal places) of the graphs of the equations. Verify your results algebraically. Test your approximations for reasonableness.

93. $y = 9 - 2x$

$y = x - 3$

95. $y = 4 - x^2$

$y = 2x - 1$

✓ 97. $y = 2x^2$

$y = x^4 - 2x^2$

94. $y = \frac{1}{3}x + 2$

$y = \frac{5}{2}x - 11$

96. $y = x^3 - 3$

$y = 5 - 2x$

98. $y = -x$

$y = 2x - x^2$

Solving a Quadratic Equation by Factoring In Exercises 99–108, solve the quadratic equation by factoring. Check your solutions in the original equation.

99. $6x^2 + 3x = 0$

101. $x^2 - 2x - 8 = 0$

103. $3 + 5x - 2x^2 = 0$

105. $x^2 + 4x = 12$

107. $(x + a)^2 - b^2 = 0$

100. $9x^2 - 1 = 0$

102. $x^2 - 10x + 9 = 0$

104. $2x^2 = 19x + 33$

106. $-x^2 + 8x = 12$

108. $x^2 + 2ax + a^2 = 0$

Solving an Equation by Extracting Square Roots In Exercises 109–118, solve the equation by extracting square roots. List both the exact solutions and the decimal solutions rounded to two decimal places.

109. $x^2 = 49$

111. $(x - 12)^2 = 16$

113. $(3x - 1)^2 + 6 = 0$

115. $(2x - 1)^2 = 12$

117. $(x - 7)^2 = (x + 3)^2$

110. $x^2 = 144$

112. $(x - 5)^2 = 25$

114. $(2x + 3)^2 + 25 = 0$

116. $(4x + 7)^2 = 44$

118. $(x + 5)^2 = (x + 4)^2$

Completing the Square In Exercises 119–128, solve the quadratic equation by completing the square. Verify your answer graphically.

119. $x^2 + 4x - 32 = 0$

121. $x^2 + 6x + 2 = 0$

123. $9x^2 - 18x + 3 = 0$

125. $-6 + 2x - x^2 = 0$

127. $2x^2 + 5x - 8 = 0$

120. $x^2 - 2x - 3 = 0$

122. $x^2 + 8x + 14 = 0$

124. $4x^2 - 4x - 99 = 0$

126. $-x^2 + x - 1 = 0$

128. $9x^2 - 12x - 14 = 0$

Using the Quadratic Formula In Exercises 129–138, use the Quadratic Formula to solve the equation. Use a graphing utility to verify your solutions graphically.

129. $2 + 2x - x^2 = 0$

131. $x^2 + 8x - 4 = 0$

133. $x^2 + 3x = -8$

135. $28x - 49x^2 = 4$

130. $x^2 - 10x + 22 = 0$

132. $4x^2 - 4x - 4 = 0$

134. $x^2 + 16 = -5x$

136. $9x^2 + 24x + 16 = 0$

137. $3x^2 + 16x + 17 = 0$

138. $9x^2 - 6x - 37 = 0$

Choose the Best Method In Exercises 139–148, solve the equation using any convenient method. Use a graphing utility to verify your solutions graphically.

139. $x^2 - 2x - 1 = 0$

141. $(x + 3)^2 = 81$

143. $x^2 - 14x + 49 = 0$

145. $x^2 - x - \frac{11}{4} = 0$

147. $(x + 1)^2 = x^2$

140. $11x^2 + 33x = 0$

142. $(x - 1)^2 = -1$

144. $x^2 - 2x + \frac{13}{4} = 0$

146. $x^2 + 3x - \frac{3}{4} = 0$

148. $a^2x^2 - b^2 = 0, a \neq 0$

Solving Equations Algebraically In Exercises 149–166, find all solutions of the equation algebraically. Use a graphing utility to verify the solutions graphically.

149. $4x^4 - 16x^2 = 0$

151. $5x^3 + 30x^2 + 45x = 0$

152. $9x^4 - 24x^3 + 16x^2 = 0$

153. $4x^4 - 18x^2 = 0$

✓ 155. $x^3 - 3x^2 - x + 3 = 0$

156. $x^4 + 2x^3 - 8x - 16 = 0$

✓ 157. $x^4 - 4x^2 + 3 = 0$

159. $4x^4 - 65x^2 + 16 = 0$

161. $\frac{1}{t^2} + \frac{8}{t} + 15 = 0$

163. $6\left(\frac{s}{s+1}\right)^2 + 5\left(\frac{s}{s+1}\right) - 6 = 0$

164. $8\left(\frac{t}{t-1}\right)^2 - 2\left(\frac{t}{t-1}\right) - 3 = 0$

165. $2x + 9\sqrt{x} - 5 = 0$

166. $6x - 7\sqrt{x} - 3 = 0$

Solving Equations Algebraically In Exercises 167–186, find all solutions of the equation algebraically. Check your solutions both algebraically and graphically.

167. $\sqrt{x - 10} - 4 = 0$

✓ 169. $\sqrt{x + 1} - 3x = 1$

171. $\sqrt[3]{2x + 1} + 8 = 0$

173. $\sqrt{x} - \sqrt{x - 5} = 1$

✓ 175. $(x - 5)^{2/3} = 16$

177. $3x(x - 1)^{1/2} + 2(x - 1)^{3/2} = 0$

178. $4x^2(x - 1)^{1/3} + 6x(x - 1)^{4/3} = 0$

✓ 179. $\frac{1}{x} - \frac{1}{x + 1} = 3$

181. $x = \frac{3}{x} + \frac{1}{2}$

183. $|2x - 1| = 5$

✓ 185. $|x| = x^2 + x - 3$

168. $\sqrt{2x + 5} + 3 = 0$

170. $\sqrt{x + 5} - 2x = 3$

172. $\sqrt[3]{4x - 3} + 2 = 0$

174. $\sqrt{x} + \sqrt{x - 20} = 10$

176. $(x^2 - x - 22)^{4/3} = 16$

177. $3x(x - 1)^{1/2} + 2(x - 1)^{3/2} = 0$

178. $4x^2(x - 1)^{1/3} + 6x(x - 1)^{4/3} = 0$

180. $\frac{x}{x^2 - 4} + \frac{1}{x + 2} = 3$

182. $4x + 1 = \frac{3}{x}$

184. $|3x + 2| = 7$

186. $|x - 10| = x^2 - 10x$

Comparing x -Intercepts and Solutions of an Equation In Exercises 187–194, (a) use a graphing utility to graph the equation, (b) use the graph to approximate any x -intercepts of the graph, (c) set $y = 0$ and solve the resulting equation, and (d) compare the result of part (c) with the x -intercepts of the graph.

$$\begin{array}{ll} 187. y = x^3 - 2x^2 - 3x & 188. y = x^4 - 10x^2 + 9 \\ 189. y = \sqrt{11x - 30} - x & 190. y = 2x - \sqrt{15 - 4x} \\ 191. y = \frac{1}{x} - \frac{4}{x-1} - 1 & 192. y = x + \frac{9}{x+1} - 5 \\ 193. y = |x + 1| - 2 & 194. y = |x - 2| - 3 \end{array}$$

195. Why you should learn it (p. C19) The populations (in thousands) of Maryland M and Arizona A from 1990 through 2008 can be modeled by

$$M = 51.1t + 4785, \quad 0 \leq t \leq 18$$

$$A = 162.0t + 3522, \quad 0 \leq t \leq 18$$

where t represents the year, with $t = 0$ corresponding to 1990. (Source: U.S. Census Bureau)

- Use a graphing utility to graph each model in the same viewing window over the appropriate domain. Approximate the point of intersection. Round your result to one decimal place. Explain the meaning of the coordinates of the point.
 - Find the point of intersection algebraically. Round your result to one decimal place. What does the point of intersection represent?
 - Explain the meaning of the slopes of both models and what they tell you about the population growth rates.
 - Use the models to estimate the population of each state in 2014. Do the values seem reasonable? Explain.
- 196. Education** The average salaries S (in thousands of dollars) of secondary classroom teachers in the United States from 2000 through 2007 can be approximated by the model
- $$S = 0.010t^2 + 1.01t + 42.9, \quad 0 \leq t \leq 7$$
- where t represents the year, with $t = 0$ corresponding to 2000. (Source: National Education Association)
- Determine algebraically when the average salary of a secondary classroom teacher was \$47,000.
 - Verify your answer to part (a) by creating a table of values for the model.
 - Use a graphing utility to graph the model.
 - Use the model to predict when the average salary will reach \$58,000.
 - Do you believe the model could be used to predict the average salaries for years beyond 2007? Explain your reasoning.

197. Biology The metabolic rate of an ectothermic organism increases with increasing temperature within a certain range. Experimental data for oxygen consumption C (in microliters per gram per hour) of a beetle at certain temperatures yielded the model

$$C = 0.45x^2 - 1.65x + 50.75, \quad 10 \leq x \leq 25$$

where x is the air temperature in degrees Celsius.

- Use a graphing utility to graph the consumption model over the specified domain.
- Use the graph to approximate the air temperature resulting in oxygen consumption of 150 microliters per gram per hour.
- The temperature is increased from 10°C to 20°C . The oxygen consumption is increased by approximately what factor?

198. Thermodynamics The temperature T (in degrees Fahrenheit) of saturated steam increases as pressure increases. This relationship is approximated by

$$T = 75.82 - 2.11x + 43.51\sqrt{x}, \quad 5 \leq x \leq 40$$

where x is the absolute pressure in pounds per square inch.

- Use a graphing utility to graph the model over the specified domain.
- The temperature of steam at sea level ($x = 14.696$) is 212°F . Evaluate the model at this pressure and verify the result graphically.
- Use the model to approximate the pressure for a steam temperature of 240°F .

Conclusions

True or False? In Exercises 199 and 200, determine whether the statement is true or false. Justify your answer.

- Two linear equations can have either one point of intersection or no points of intersection.
- An equation can never have more than one extraneous solution.
- Think About It** Find c such that $x = 3$ is a solution to the linear equation $2x - 5c = 10 + 3c - 3x$.
- Think About It** Find c such that $x = 2$ is a solution to the linear equation $5x + 2c = 12 + 4x - 2c$.
- Exploration** Given that a and b are nonzero real numbers, determine the solutions of the equations.
 - $ax^2 + bx = 0$
 - $ax^2 - ax = 0$

C.4 Solving Inequalities Algebraically and Graphically

Properties of Inequalities

The inequality symbols

$<$, \leq , $>$, and \geq *Inequality symbols*

are used to compare two numbers and to denote subsets of real numbers. For instance, the simple inequality $x \geq 3$ denotes all real numbers x that are greater than or equal to 3.

In this appendix, you will study inequalities that contain more involved statements such as

$$5x - 7 > 3x + 9$$

and

$$-3 \leq 6x - 1 < 3.$$

As with an equation, you **solve an inequality** in the variable x by finding all values of x for which the inequality is true. These values are **solutions** of the inequality and are said to **satisfy** the inequality. For instance, the number 9 is a solution of the first inequality listed above because

$$5(9) - 7 > 3(9) + 9$$

$$45 - 7 > 27 + 9$$

$$38 > 36.$$

On the other hand, the number 7 is not a solution because

$$5(7) - 7 \not> 3(7) + 9$$

$$35 - 7 \not> 21 + 9$$

$$28 \not> 30.$$

The set of all real numbers that are solutions of an inequality is the **solution set** of the inequality.

The set of all points on the real number line that represent the solution set is the **graph of the inequality**. Graphs of many types of inequalities consist of intervals on the real number line.

The procedures for solving linear inequalities in one variable are much like those for solving linear equations. To isolate the variable, you can make use of the **properties of inequalities**. These properties are similar to the properties of equality, but there are two important exceptions. When each side of an inequality is multiplied or divided by a negative number, *the direction of the inequality symbol must be reversed* in order to maintain a true statement. Here is an example.

$$-2 < 5$$

Original inequality

$$(-3)(-2) > (-3)(5)$$

Multiply each side by -3 and reverse the inequality.

$$6 > -15$$

Simplify.

Two inequalities that have the same solution set are **equivalent inequalities**.

For instance, the inequalities

$$x + 2 < 5 \quad \text{and} \quad x < 3$$

are equivalent. To obtain the second inequality from the first, you can subtract 2 from each side of the inequality. The properties listed at the top of the next page describe operations that can be used to create equivalent inequalities.

What you should learn

- Use properties of inequalities to solve linear inequalities.
- Solve inequalities involving absolute values.
- Solve polynomial inequalities.
- Solve rational inequalities.
- Use inequalities to model and solve real-life problems.

Why you should learn it

An inequality can be used to determine when a real-life quantity exceeds a given level. For instance, Exercises 99–102 on page C47 show how to use linear inequalities to determine when the number of Bed Bath & Beyond stores exceeds the number of Williams-Sonoma stores.

Properties of Inequalities

Let a , b , c , and d be real numbers.

1. Transitive Property

$$a < b \text{ and } b < c \quad \Rightarrow \quad a < c$$

2. Addition of Inequalities

$$a < b \text{ and } c < d \quad \Rightarrow \quad a + c < b + d$$

3. Addition of a Constant

$$a < b \quad \Rightarrow \quad a + c < b + c$$

4. Multiplying by a Constant

$$\text{For } c > 0, a < b \quad \Rightarrow \quad ac < bc$$

$$\text{For } c < 0, a < b \quad \Rightarrow \quad ac > bc$$

Each of the properties above is true when the symbol $<$ is replaced by \leq and $>$ is replaced by \geq . For instance, another form of Property 3 is as follows.

$$a \leq b \quad \Rightarrow \quad a + c \leq b + c$$

The simplest type of inequality to solve is a **linear inequality** in one variable, such as $2x + 3 > 4$. (For help with solving one-step linear inequalities, see Appendix E at this textbook's *Companion Website*.)

Example 1 Solving a Linear Inequality

Solve $5x - 7 > 3x + 9$.

Solution

$$5x - 7 > 3x + 9 \quad \text{Original inequality}$$

$$2x - 7 > 9 \quad \text{Subtract } 3x \text{ from each side.}$$

$$2x > 16 \quad \text{Add 7 to each side.}$$

$$x > 8 \quad \text{Divide each side by 2.}$$

So, the solution set is all real numbers that are greater than 8. The interval notation for this solution set is

$$(8, \infty).$$

The graph of this solution set is shown in Figure C.44. Note that a parenthesis at 8 on the number line indicates that 8 is *not* part of the solution set.

CHECKPOINT Now try Exercise 19.

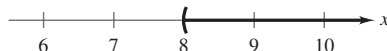


Figure C.44 Solution Interval: $(8, \infty)$

Explore the Concept

Use a graphing utility to graph $f(x) = 5x - 7$ and $g(x) = 3x + 9$ in the same viewing window. (Use $-1 \leq x \leq 15$ and $-5 \leq y \leq 50$.) For which values of x does the graph of f lie above the graph of g ? Explain how the answer to this question can be used to solve the inequality in Example 1.

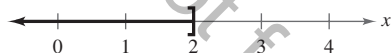
Note that the four inequalities forming the solution steps of Example 1 are all *equivalent* in the sense that each has the same solution set.

Checking the solution set of an inequality is not as simple as checking the solution(s) of an equation because there are simply too many x -values to substitute into the original inequality. However, you can get an indication of the validity of the solution set by substituting a few convenient values of x . For instance, in Example 1, try substituting $x = 6$ and $x = 10$ into the original inequality.

Example 2 Solving an InequalitySolve $1 - \frac{3}{2}x \geq x - 4$.**Algebraic Solution**

$$\begin{array}{ll}
 1 - \frac{3}{2}x \geq x - 4 & \text{Write original inequality.} \\
 2 - 3x \geq 2x - 8 & \text{Multiply each side by the LCD.} \\
 2 - 5x \geq -8 & \text{Subtract } 2x \text{ from each side.} \\
 -5x \geq -10 & \text{Subtract 2 from each side.} \\
 x \leq 2 & \text{Divide each side by } -5 \\
 & \text{and reverse the inequality.}
 \end{array}$$

The solution set is all real numbers that are less than or equal to 2. The interval notation for this solution set is $(-\infty, 2]$. The graph of this solution set is shown in Figure C.45. Note that a bracket at 2 on the number line indicates that 2 is part of the solution set.

**Figure C.45** Solution Interval: $(-\infty, 2]$

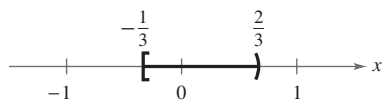
CHECKPOINT Now try Exercise 23.

Sometimes it is possible to write two inequalities as a **double inequality**, as demonstrated in Example 3.

Example 3 Solving a Double InequalitySolve $-3 \leq 6x - 1$ and $6x - 1 < 3$.**Algebraic Solution**

$$\begin{array}{ll}
 -3 \leq 6x - 1 < 3 & \text{Write as a double inequality.} \\
 -3 + 1 \leq 6x - 1 + 1 < 3 + 1 & \text{Add 1 to each part.} \\
 -2 \leq 6x < 4 & \text{Simplify.} \\
 \frac{-2}{6} \leq \frac{6x}{6} < \frac{4}{6} & \text{Divide each part by 6.} \\
 -\frac{1}{3} \leq x < \frac{2}{3} & \text{Simplify.}
 \end{array}$$

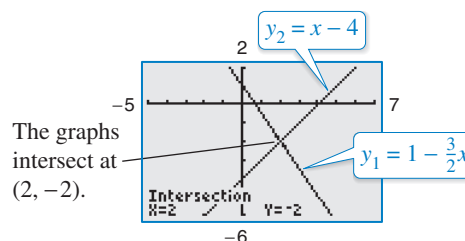
The solution set is all real numbers that are greater than or equal to $-\frac{1}{3}$ and less than $\frac{2}{3}$. The interval notation for this solution set is $[-\frac{1}{3}, \frac{2}{3})$. The graph of this solution set is shown in Figure C.47.

**Figure C.47** Solution Interval: $[-\frac{1}{3}, \frac{2}{3})$

CHECKPOINT Now try Exercise 25.

Graphical Solution

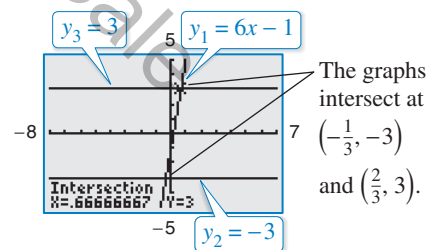
Use a graphing utility to graph $y_1 = 1 - \frac{3}{2}x$ and $y_2 = x - 4$ in the same viewing window, as shown in Figure C.46.

**Figure C.46**

The graph of y_1 lies above the graph of y_2 to the left of their point of intersection, $(2, -2)$, which implies that $y_1 \geq y_2$ for all $x \leq 2$.

Graphical Solution

Use a graphing utility to graph $y_1 = 6x - 1$, $y_2 = -3$, and $y_3 = 3$ in the same viewing window, as shown in Figure C.48.

**Figure C.48**

The graph of y_1 lies above the graph of y_2 to the right of $(-\frac{1}{3}, -3)$ and the graph of y_1 lies below the graph of y_3 to the left of $(\frac{2}{3}, 3)$. This implies that $y_2 \leq y_1 < y_3$ when $-\frac{1}{3} \leq x < \frac{2}{3}$.

Inequalities Involving Absolute Values

Solving an Absolute Value Inequality

Let x be a variable or an algebraic expression and let a be a real number such that $a \geq 0$.

1. The solutions of $|x| < a$ are all values of x that lie between $-a$ and a .

$$|x| < a \text{ if and only if } -a < x < a. \quad \text{Double inequality}$$

2. The solutions of $|x| > a$ are all values of x that are less than $-a$ or greater than a .

$$|x| > a \text{ if and only if } x < -a \text{ or } x > a. \quad \text{Compound inequality}$$

These rules are also valid when $<$ is replaced by \leq and $>$ is replaced by \geq .

Example 4 Solving Absolute Value Inequalities

Solve each inequality.

a. $|x - 5| < 2$ b. $|x - 5| > 2$

Algebraic Solution

a. $|x - 5| < 2$

Write original inequality.

$$-2 < x - 5 < 2$$

Write double inequality.

$$3 < x < 7$$

Add 5 to each part.

The solution set is all real numbers that are greater than 3 and less than 7. The interval notation for this solution set is $(3, 7)$. The graph of this solution set is shown in Figure C.49.

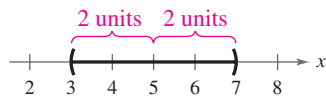


Figure C.49

- b. The absolute value inequality $|x - 5| > 2$ is equivalent to the following compound inequality: $x - 5 < -2$ or $x - 5 > 2$.

Solve first inequality: $x - 5 < -2$

Write first inequality.

$$x < 3$$

Add 5 to each side.

Solve second inequality: $x - 5 > 2$

Write second inequality.

$$x > 7$$

Add 5 to each side.

The solution set is all real numbers that are less than 3 or greater than 7. The interval notation for this solution set is $(-\infty, 3) \cup (7, \infty)$. The symbol \cup is called a *union* symbol and is used to denote the combining of two sets. The graph of this solution set is shown in Figure C.50.

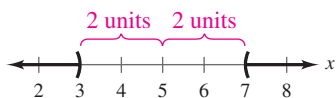


Figure C.50

Graphical Solution

Use a graphing utility to graph

$$y_1 = |x - 5| \text{ and } y_2 = 2$$

in the same viewing window, as shown in Figure C.51.

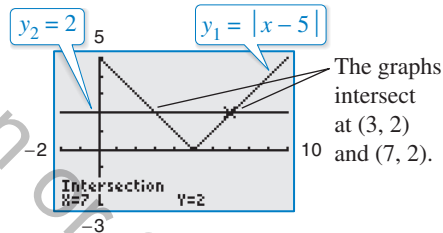


Figure C.51

- a. In Figure C.51, you can see that the graph of y_1 lies below the graph of y_2 when

$$3 < x < 7.$$

This implies that the solution set is all real numbers greater than 3 and less than 7.

- b. In Figure C.51, you can see that the graph of y_1 lies above the graph of y_2 when

$$x < 3$$

or when

$$x > 7.$$

This implies that the solution set is all real numbers that are less than 3 or greater than 7.

Polynomial Inequalities

To solve a polynomial inequality such as $x^2 - 2x - 3 < 0$, use the fact that a polynomial can change signs only at its zeros (the x -values that make the polynomial equal to zero). Between two consecutive zeros, a polynomial must be entirely positive or entirely negative. This means that when the real zeros of a polynomial are put in order, they divide the real number line into intervals in which the polynomial has no sign changes. These zeros are the **key numbers** of the inequality, and the resulting open intervals are the **test intervals** for the inequality. For instance, the polynomial above factors as

$$x^2 - 2x - 3 = (x + 1)(x - 3)$$

and has two zeros, $x = -1$ and $x = 3$, which divide the real number line into three test intervals: $(-\infty, -1)$, $(-1, 3)$, and $(3, \infty)$. To solve the inequality $x^2 - 2x - 3 < 0$, you need to test only one value in each test interval.

Finding Test Intervals for a Polynomial

To determine the intervals on which the values of a polynomial are entirely negative or entirely positive, use the following steps.

1. Find all real zeros of the polynomial, and arrange the zeros in increasing order. These zeros are the key numbers of the polynomial.
2. Use the key numbers to determine the test intervals.
3. Choose one representative x -value in each test interval and evaluate the polynomial at that value. If the value of the polynomial is negative, then the polynomial will have negative values for *every* x -value in the interval. If the value of the polynomial is positive, then the polynomial will have positive values for *every* x -value in the interval.

Example 5 Investigating Polynomial Behavior

To determine the intervals on which $x^2 - 3$ is entirely negative and those on which it is entirely positive, factor the quadratic as

$$x^2 - 3 = (x + \sqrt{3})(x - \sqrt{3}).$$

The key numbers occur at $x = -\sqrt{3}$ and $x = \sqrt{3}$. So, the test intervals for the quadratic are

$$(-\infty, -\sqrt{3}), \quad (-\sqrt{3}, \sqrt{3}), \quad \text{and} \quad (\sqrt{3}, \infty).$$

In each test interval, choose a representative x -value and evaluate the polynomial, as shown in the table.

Interval	x -Value	Value of Polynomial	Sign of Polynomial
$(-\infty, -\sqrt{3})$	$x = -3$	$(-3)^2 - 3 = 6$	Positive
$(-\sqrt{3}, \sqrt{3})$	$x = 0$	$(0)^2 - 3 = -3$	Negative
$(\sqrt{3}, \infty)$	$x = 5$	$(5)^2 - 3 = 22$	Positive

The polynomial has negative values for every x in the interval $(-\sqrt{3}, \sqrt{3})$ and positive values for every x in the intervals $(-\infty, -\sqrt{3})$ and $(\sqrt{3}, \infty)$. This result is shown graphically in Figure C.52.

 **CHECKPOINT** Now try Exercise 53.

Technology Tip



Some graphing utilities will produce graphs of inequalities. For

instance, you can graph $2x^2 + 5x > 12$ by setting the graphing utility to *dot* mode and entering $y = 2x^2 + 5x > 12$.

Using $-10 \leq x \leq 10$ and $-4 \leq y \leq 4$, your graph should look like the graph shown below. The solution appears to be $(-\infty, -4) \cup (\frac{3}{2}, \infty)$. See Example 6 for an algebraic solution, and for an alternative graphical solution.

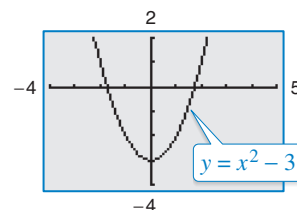
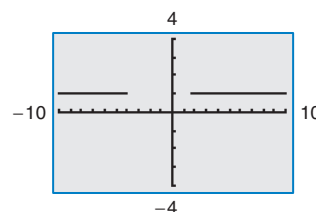


Figure C.52

To determine the test intervals for a polynomial inequality, the inequality must first be written in general form with the polynomial on one side and zero on the other.

Example 6 Solving a Polynomial Inequality

Solve $2x^2 + 5x > 12$.

Algebraic Solution

$$2x^2 + 5x - 12 > 0 \quad \text{Write inequality in general form.}$$

$$(x + 4)(2x - 3) > 0 \quad \text{Factor.}$$

Key Numbers: $x = -4, x = \frac{3}{2}$

Test Intervals: $(-\infty, -4), (-4, \frac{3}{2}), (\frac{3}{2}, \infty)$

Test: Is $(x + 4)(2x - 3) > 0$?

After testing these intervals, you can see that the polynomial $2x^2 + 5x - 12$ is positive on the open intervals $(-\infty, -4)$ and $(\frac{3}{2}, \infty)$. Therefore, the solution set of the inequality is

$$(-\infty, -4) \cup (\frac{3}{2}, \infty).$$

 **CHECKPOINT** Now try Exercise 65.

Graphical Solution

First write the polynomial inequality $2x^2 + 5x > 12$ as $2x^2 + 5x - 12 > 0$. Then use a graphing utility to graph $y = 2x^2 + 5x - 12$. In Figure C.53, you can see that the graph is *above* the x -axis when x is less than -4 or when x is greater than $\frac{3}{2}$. So, you can graphically approximate the solution set to be $(-\infty, -4) \cup (\frac{3}{2}, \infty)$.

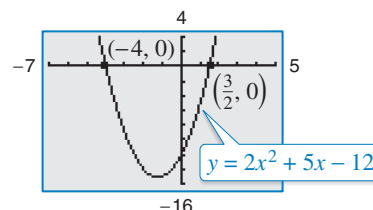


Figure C.53

Example 7 Solving a Polynomial Inequality

$$2x^3 - 3x^2 - 32x + 48 > 0 \quad \text{Original inequality}$$

$$x^2(2x - 3) - 16(2x - 3) > 0 \quad \text{Factor by grouping.}$$

$$(x^2 - 16)(2x - 3) > 0 \quad \text{Distributive Property}$$

$$(x - 4)(x + 4)(2x - 3) > 0 \quad \text{Factor difference of two squares.}$$

The key numbers are $x = -4, x = \frac{3}{2}$, and $x = 4$; and the test intervals are $(-\infty, -4)$, $(-4, \frac{3}{2})$, $(\frac{3}{2}, 4)$, and $(4, \infty)$.

Interval	x -Value	Polynomial Value	Conclusion
$(-\infty, -4)$	$x = -5$	$2(-5)^3 - 3(-5)^2 - 32(-5) + 48 = -117$	Negative
$(-4, \frac{3}{2})$	$x = 0$	$2(0)^3 - 3(0)^2 - 32(0) + 48 = 48$	Positive
$(\frac{3}{2}, 4)$	$x = 2$	$2(2)^3 - 3(2)^2 - 32(2) + 48 = -12$	Negative
$(4, \infty)$	$x = 5$	$2(5)^3 - 3(5)^2 - 32(5) + 48 = 63$	Positive

From this you can conclude that the polynomial is positive on the open intervals $(-4, \frac{3}{2})$ and $(4, \infty)$. So, the solution set is

$$(-4, \frac{3}{2}) \cup (4, \infty).$$

 **CHECKPOINT** Now try Exercise 69.

When solving a polynomial inequality, be sure you have accounted for the particular type of inequality symbol given in the inequality. For instance, in Example 7, note that the original inequality contained a “greater than” symbol and the solution consisted of two open intervals. If the original inequality had been

$$2x^3 - 3x^2 - 32x + 48 \geq 0$$

the solution would have consisted of the closed interval $[-4, \frac{3}{2}]$ and the interval $[4, \infty)$.

Example 8 Unusual Solution Sets

- a. The solution set of

$$x^2 + 2x + 4 > 0$$

consists of the entire set of real numbers, $(-\infty, \infty)$. In other words, the value of the quadratic $x^2 + 2x + 4$ is positive for every real value of x , as indicated in Figure C.54. (Note that this quadratic inequality has *no* key numbers. In such a case, there is only one test interval—the entire real number line.)

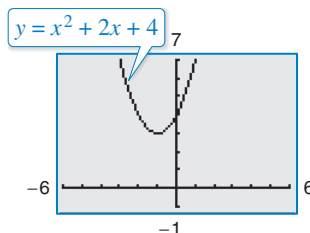


Figure C.54

- b. The solution set of

$$x^2 + 2x + 1 \leq 0$$

consists of the single real number $\{-1\}$, because the quadratic

$$x^2 + 2x + 1$$

has one key number, $x = -1$, and it is the only value that satisfies the inequality, as indicated in Figure C.55.

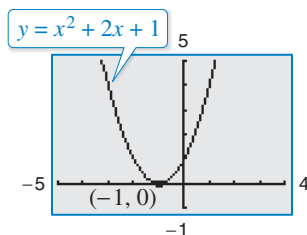


Figure C.55

- c. The solution set of

$$x^2 + 3x + 5 < 0$$

is empty. In other words, the quadratic

$$x^2 + 3x + 5$$

is not less than zero for any value of x , as indicated in Figure C.56.

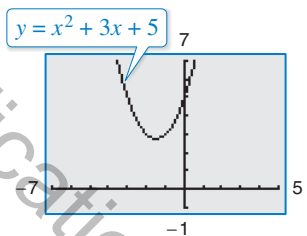


Figure C.56

- d. The solution set of

$$x^2 - 4x + 4 > 0$$

consists of all real numbers *except* the number 2. In interval notation, this solution set can be written as $(-\infty, 2) \cup (2, \infty)$.

The graph of $y = x^2 - 4x + 4$ lies above the x -axis except at $x = 2$, where it touches it, as indicated in Figure C.57.

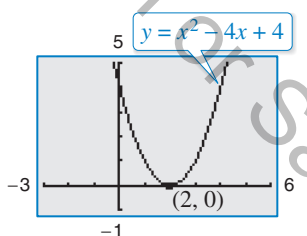


Figure C.57

CHECKPOINT Now try Exercise 73.

Technology Tip

One of the advantages of technology is that you can solve complicated polynomial inequalities that might be difficult, or even impossible, to factor. For instance, you could use a graphing utility to approximate the solution of the inequality

$$x^3 - 0.2x^2 - 3.16x + 1.4 < 0.$$

Rational Inequalities

The concepts of key numbers and test intervals can be extended to inequalities involving rational expressions. To do this, use the fact that the value of a rational expression can change sign only at its *zeros* (the x -values for which its numerator is zero) and its *undefined values* (the x -values for which its denominator is zero). These two types of numbers make up the *key numbers* of a rational inequality. When solving a rational inequality, begin by writing the inequality in general form with the rational expression on one side and zero on the other.

Example 9 Solving a Rational Inequality

Solve $\frac{2x - 7}{x - 5} \leq 3$.

Algebraic Solution

$$\frac{2x - 7}{x - 5} - 3 \leq 0 \quad \text{Write in general form.}$$

$$\frac{2x - 7 - 3x + 15}{x - 5} \leq 0 \quad \text{Write as single fraction.}$$

$$\frac{-x + 8}{x - 5} \leq 0 \quad \text{Simplify.}$$

Now, in standard form you can see that the key numbers are $x = 5$ and $x = 8$, and you can proceed as follows.

Key Numbers: $x = 5, x = 8$

Test Intervals: $(-\infty, 5), (5, 8), (8, \infty)$

Test: Is $\frac{-x + 8}{x - 5} \leq 0$?

Interval	x -Value	Polynomial Value	Conclusion
$(-\infty, 5)$	$x = 0$	$\frac{-0 + 8}{0 - 5} = -\frac{8}{5}$	Negative
$(5, 8)$	$x = 6$	$\frac{-6 + 8}{6 - 5} = 2$	Positive
$(8, \infty)$	$x = 9$	$\frac{-9 + 8}{9 - 5} = -\frac{1}{4}$	Negative

By testing these intervals, you can determine that the rational expression

$$\frac{-x + 8}{x - 5}$$

is negative in the open intervals $(-\infty, 5)$ and $(8, \infty)$. Moreover, because

$$\frac{-x + 8}{x - 5} = 0$$

when $x = 8$, you can conclude that the solution set of the inequality is $(-\infty, 5) \cup [8, \infty)$.

CHECKPOINT Now try Exercise 83.

Graphical Solution

Use a graphing utility to graph

$$y_1 = \frac{2x - 7}{x - 5} \quad \text{and} \quad y_2 = 3$$

in the same viewing window, as shown in Figure C.58.

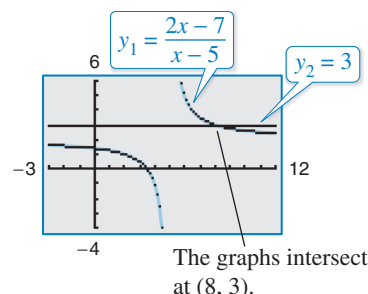


Figure C.58

The graph of y_1 lies below the graph of y_2 in the intervals $(-\infty, 5)$ and $[8, \infty)$. So, you can graphically estimate the solution set to be all real numbers less than 5 or greater than or equal to 8.

Note in Example 9 that $x = 5$ is not included in the solution set because the inequality is undefined when $x = 5$.

Applications

In Section 1.2 you studied the *implied domain* of a function, the set of all x -values for which the function is defined. A common type of implied domain is used to avoid even roots of negative numbers, as shown in Example 10.

Example 10 Finding the Domain of an Expression

Find the domain of $\sqrt{64 - 4x^2}$.

Solution

Because $\sqrt{64 - 4x^2}$ is defined only when $64 - 4x^2$ is nonnegative, the domain is given by $64 - 4x^2 \geq 0$.

$$64 - 4x^2 \geq 0 \quad \text{Write in general form.}$$

$$16 - x^2 \geq 0 \quad \text{Divide each side by 4.}$$

$$(4 - x)(4 + x) \geq 0 \quad \text{Factor.}$$

The inequality has two key numbers: $x = -4$ and $x = 4$. A test shows that $64 - 4x^2 \geq 0$ in the *closed interval* $[-4, 4]$. The graph of $y = \sqrt{64 - 4x^2}$, shown in Figure C.59, confirms that the domain is $[-4, 4]$.

 **CHECKPOINT** Now try Exercise 91.

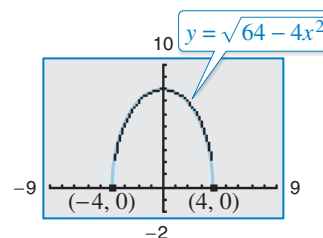


Figure C.59

Example 11 Height of a Projectile



A projectile is fired straight upward from ground level with an initial velocity of 384 feet per second. During what time period will the height of the projectile exceed 2000 feet?

Solution

The position of an object moving vertically can be modeled by the *position equation*

$$s = -16t^2 + v_0t + s_0$$

where s is the height in feet and t is the time in seconds. In this case, $s_0 = 0$ and $v_0 = 384$. So, you need to solve the inequality $-16t^2 + 384t > 2000$. Using a graphing utility, graph $y_1 = -16t^2 + 384t$ and $y_2 = 2000$, as shown in Figure C.60. From the graph, you can determine that $-16t^2 + 384t > 2000$ for t between approximately 7.6 and 16.4. You can verify this result algebraically.

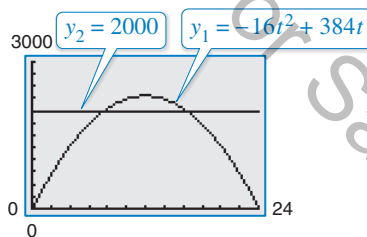


Figure C.60

$$-16t^2 + 384t > 2000 \quad \text{Write original inequality.}$$

$$t^2 - 24t < -125 \quad \text{Divide by } -16 \text{ and reverse inequality.}$$

$$t^2 - 24t + 125 < 0 \quad \text{Write in general form.}$$

By the Quadratic Formula the key numbers are $t = 12 - \sqrt{19}$ and $t = 12 + \sqrt{19}$, or approximately 7.64 and 16.36. A test will verify that the height of the projectile will exceed 2000 feet when $7.64 < t < 16.36$; that is, during the time interval $(7.64, 16.36)$ seconds.

 **CHECKPOINT** Now try Exercise 95.

C.4 Exercises

For instructions on how to use a graphing utility, see Appendix A.

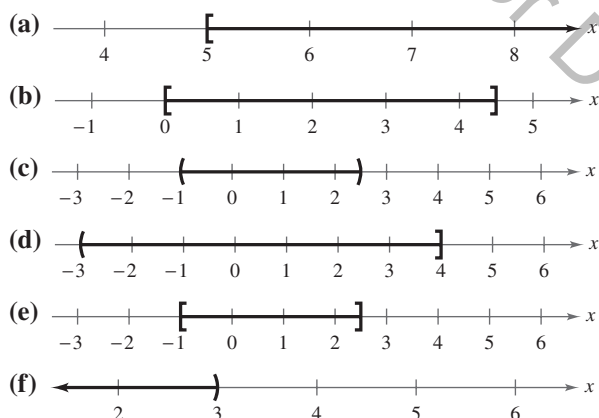
Vocabulary and Concept Check

In Exercises 1–4, fill in the blank(s).

- It is sometimes possible to write two inequalities as one inequality, called a _____ inequality.
- The solutions of $|x| \leq a$ are those values of x such that _____.
- The solutions of $|x| \geq a$ are those values of x such that _____ or _____.
- The key numbers of a rational inequality are its _____ and its _____.
- Are the inequalities $x - 4 < 5$ and $x > 9$ equivalent?
- Which property of inequalities is shown below?
 $a < b$ and $b < c \Rightarrow a < c$

Procedures and Problem Solving

Matching an Inequality with Its Graph In Exercises 7–12, match the inequality with its graph. [The graphs are labeled (a), (b), (c), (d), (e), and (f).]



- $x < 3$
- $x \geq 5$
- $-3 < x \leq 4$
- $0 \leq x \leq \frac{9}{2}$
- $-1 \leq x \leq \frac{5}{2}$
- $-1 < x < \frac{5}{2}$

Determining Solutions of an Inequality In Exercises 13–16, determine whether each value of x is a solution of the inequality.

Inequality	Values	
13. $5x - 12 > 0$	(a) $x = 3$	(b) $x = -3$
	(c) $x = \frac{5}{2}$	(d) $x = \frac{3}{2}$
14. $-5 < 2x - 1 \leq 1$	(a) $x = -\frac{1}{2}$	(b) $x = -\frac{5}{2}$
	(c) $x = \frac{4}{3}$	(d) $x = 0$
15. $-1 < \frac{3-x}{2} \leq 1$	(a) $x = -1$	(b) $x = \sqrt{5}$
	(c) $x = 1$	(d) $x = 5$
16. $ x - 10 \geq 3$	(a) $x = 13$	(b) $x = -1$
	(c) $x = 14$	(d) $x = 8$

Solving an Inequality In Exercises 17–30, solve the inequality and sketch the solution on the real number line. Use a graphing utility to verify your solution graphically.

- $-10x < 40$
- $6x > 15$
- $4x + 7 < 3 + 2x$
- $3x + 1 \geq 2 + x$
- $4(x + 1) < 2x + 3$
- $2x + 7 < 3(x - 4)$
- $\frac{3}{4}x - 6 \leq x - 7$
- $3 + \frac{2}{7}x > x - 2$
- $1 < 2x + 3 < 9$
- $-8 \leq -3x + 5 < 13$
- $-8 \leq 1 - 3(x - 2) < 13$
- $0 \leq 2 - 3(x + 1) < 20$
- $-4 < \frac{2x - 3}{3} < 4$
- $0 \leq \frac{x + 3}{2} < 5$

Approximating a Solution In Exercises 31–34, use a graphing utility to approximate the solution.

- $5 - 2x \geq 1$
- $20 < 6x - 1$
- $3(x + 1) < x + 7$
- $4(x - 3) \leq 8 - x$

Approximating Solutions In Exercises 35–38, use a graphing utility to graph the equation and graphically approximate the values of x that satisfy the specified inequalities. Then solve each inequality algebraically.

Equation	Inequalities	
35. $y = 2x - 3$	(a) $y \geq 1$	(b) $y \leq 0$
36. $y = -3x + 8$	(a) $-1 \leq y \leq 3$	(b) $y \leq 0$
37. $y = -\frac{1}{2}x + 2$	(a) $0 \leq y \leq 3$	(b) $y \geq 0$
38. $y = \frac{2}{3}x + 1$	(a) $y \leq 5$	(b) $y \geq 0$

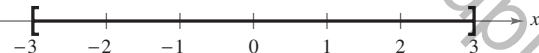
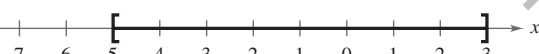
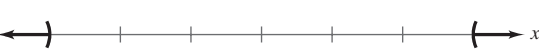

Solving an Absolute Value Inequality In Exercises 39–46, solve the inequality and sketch the solution on the real number line. Use a graphing utility to verify your solutions graphically.

39. $|5x| > 10$ 40. $\left|\frac{x}{2}\right| \leq 1$
 ✓ 41. $|x - 7| < 6$ 42. $|x - 20| \geq 4$
 43. $|x + 14| + 3 > 17$ 44. $\left|\frac{x - 3}{2}\right| \geq 5$
 45. $10|1 - x| < 5$ 46. $3|4 - 5x| \leq 9$

Approximating Solutions In Exercises 47 and 48, use a graphing utility to graph the equation and graphically approximate the values of x that satisfy the specified inequalities. Then solve each inequality algebraically.

Equation	Inequalities
47. $y = x - 3 $	(a) $y \leq 2$ (b) $y \geq 4$
48. $y = \left \frac{1}{2}x + 1\right $	(a) $y \leq 4$ (b) $y \geq 1$

Using Absolute Value Notation In Exercises 49–56, use absolute value notation to define the interval (or pair of intervals) on the real number line.

49.  x
 50.  x
 51.  x
 52.  x

53. All real numbers within 10 units of 7
 54. All real numbers no more than 8 units from -5
 55. All real numbers at least 5 units from 3
 56. All real numbers more than 3 units from -1

Investigating Polynomial Behavior In Exercises 57–62, determine the intervals on which the polynomial is entirely negative and those on which it is entirely positive.

57. $x^2 - 4x - 5$ 58. $x^2 - 3x - 4$
 ✓ 59. $2x^2 - 4x - 3$ 60. $2x^2 - x - 5$
 61. $x^2 - 4x + 5$ 62. $-x^2 + 6x - 10$

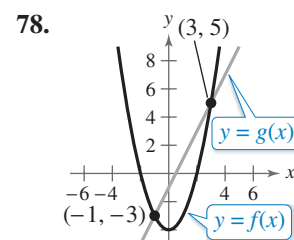
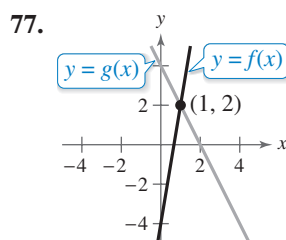
Solving a Polynomial Inequality In Exercises 63–76, solve the inequality and graph the solution on the real number line. Use a graphing utility to verify your solution graphically.

63. $(x + 2)^2 < 25$ 64. $(x - 3)^2 \geq 1$
 ✓ 65. $x^2 + 4x + 4 \geq 9$ 66. $x^2 - 6x + 9 < 16$

67. $x^3 - 4x \geq 0$ 68. $x^4(x - 3) \leq 0$
 ✓ 69. $2x^3 + 5x^2 > 6x + 9$ 70. $2x^3 + 3x^2 < 11x + 6$
 71. $x^3 - 3x^2 - x > -3$
 72. $2x^3 + 13x^2 - 8x - 46 \geq 6$
 ✓ 73. $3x^2 - 11x + 16 \leq 0$ 74. $4x^2 + 12x + 9 \leq 0$
 75. $x^2 + 3x + 8 > 0$ 76. $4x^2 - 4x + 1 \leq 0$

Using Graphs to Find Solutions In Exercises 77 and 78, use the graph of the function to solve the equation or inequality.

- (a) $f(x) = g(x)$ (b) $f(x) \geq g(x)$ (c) $f(x) > g(x)$



Approximating Solutions In Exercises 79 and 80, use a graphing utility to graph the equation and graphically approximate the values of x that satisfy the specified inequalities. Then solve each inequality algebraically.

Equation	Inequalities
79. $y = -x^2 + 2x + 3$	(a) $y \leq 0$ (b) $y \geq 3$
80. $y = x^3 - x^2 - 16x + 16$	(a) $y \leq 0$ (b) $y \geq 36$

Solving a Rational Inequality In Exercises 81–84, solve the inequality and graph the solution on the real number line. Use a graphing utility to verify your solution graphically.

81. $\frac{1}{x} - x > 0$
 82. $\frac{1}{x} - 4 < 0$
 ✓ 83. $\frac{x + 6}{x + 1} - 2 < 0$
 84. $\frac{x + 12}{x + 2} - 3 \geq 0$

Approximating Solutions In Exercises 85 and 86, use a graphing utility to graph the equation and graphically approximate the values of x that satisfy the specified inequalities. Then solve each inequality algebraically.

Equation	Inequalities
85. $y = \frac{3x}{x - 2}$	(a) $y \leq 0$ (b) $y \geq 6$
86. $y = \frac{5x}{x^2 + 4}$	(a) $y \geq 1$ (b) $y \leq 0$

Finding the Domain of an Expression In Exercises 87–92, find the domain of x in the expression.

87. $\sqrt{x-5}$

88. $\sqrt[4]{6x+15}$

89. $\sqrt[3]{6-x}$

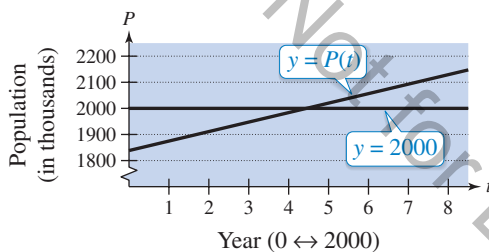
90. $\sqrt[3]{2x^2-8}$

✓ 91. $\sqrt{x^2-4}$

92. $\sqrt[4]{4-x^2}$

93. MODELING DATA

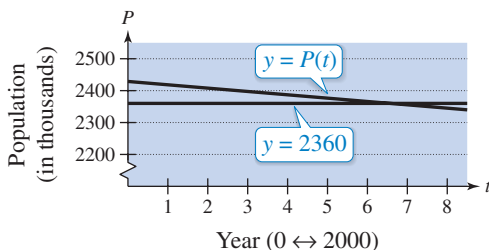
The graph models the population P (in thousands) of Sacramento, California from 2000 through 2008, where t is the year, with $t = 0$ corresponding to 2000. Also shown is the line $y = 2000$. Use the graphs of the model and the horizontal line to write an equation or an inequality that could be solved to answer the question. Then answer the question. (Source: U.S. Census Bureau)



- In what year did the population of Sacramento reach two million?
- Over what time period is the population of Sacramento less than two million? greater than two million?

94. MODELING DATA

The graph models the population P (in thousands) of Pittsburgh, Pennsylvania from 2000 through 2008, where t is the year, with $t = 0$ corresponding to 2000. Also shown is the line $y = 2360$. Use the graphs of the model and the horizontal line to write an equation or an inequality that could be solved to answer the question. Then answer the question. (Source: U.S. Census Bureau)



- In what year did the population of Pittsburgh reach 2.36 million?
- Over what time period is the population of Pittsburgh less than 2.36 million? greater than 2.36 million?

✓ 95. **Height of a Projectile** A projectile is fired straight upward from ground level with an initial velocity of 160 feet per second.

- At what instant will it be back at ground level?
- When will the height exceed 384 feet?

96. **Height of a Projectile** A projectile is fired straight upward from ground level with an initial velocity of 128 feet per second.

- At what instant will it be back at ground level?
- When will the height be less than 128 feet?

97. MODELING DATA

The numbers D of doctorate degrees (in thousands) awarded to female students from 1990 through 2008 in the United States can be approximated by the model

$$D = 0.0510t^2 - 0.045t + 15.25, \quad 0 \leq t \leq 18$$

where t is the year, with $t = 0$ corresponding to 1990. (Source: U.S. National Center for Education Statistics)

- Use a graphing utility to graph the model.
- Use the *zoom* and *trace* features to find when the number of degrees was between 20 and 25 thousand.
- Algebraically verify your results from part (b).

98. MODELING DATA

You want to determine whether there is a relationship between an athlete's weight x (in pounds) and the athlete's maximum bench-press weight y (in pounds). Sample data from 12 athletes are shown below.

(165, 170), (184, 185), (150, 200),
(210, 255), (196, 205), (240, 295),
(202, 190), (170, 175), (185, 195),
(190, 185), (230, 250), (160, 150)

- Use a graphing utility to plot the data.
- A model for the data is

$$y = 1.3x - 36.$$

Use the graphing utility to graph the equation in the same viewing window used in part (a).

- Use the graph to estimate the value of x that predicts a maximum bench-press weight of at least 200 pounds.
- Use the graph to write a statement about the accuracy of the model. If you think the graph indicates that an athlete's weight is not a good indicator of the athlete's maximum bench-press weight, list other factors that might influence an individual's maximum bench-press weight.

Why you should learn it (p. C35) In Exercises 99–102, use the models below, which approximate the numbers of Bed Bath & Beyond stores B and Williams-Sonoma stores W for the years 2000 through 2008, where t is the year, with $t = 0$ corresponding to 2000. (Source: Bed Bath & Beyond, Inc. and Williams-Sonoma, Inc.)

Bed Bath & Beyond: $B = 91.88t + 331.7$, $0 \leq t \leq 8$

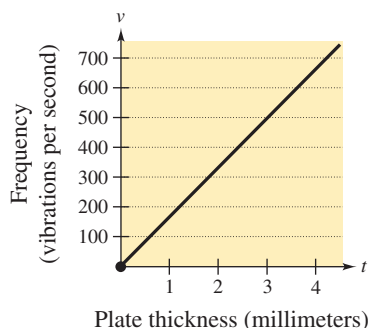
Williams-Sonoma: $W = 30.22t + 404.0$, $0 \leq t \leq 8$

99. Solve the inequality $B(t) \geq 900$. Explain what the solution of the inequality represents.
100. Solve the inequality $W(t) \leq 600$. Explain what the solution of the inequality represents.
101. Solve the equation $B(t) = W(t)$. Explain what the solution of the equation represents.
102. Solve the inequality $B(t) \geq W(t)$. Explain what the solution of the inequality represents.

Music In Exercises 103–106, use the following information. Michael Kasha of Florida State University used physics and mathematics to design a classical guitar. He used the model for the frequency of the vibrations on a circular plate

$$v = \frac{2.6t}{d^2} \sqrt{\frac{E}{\rho}}$$

where v is the frequency (in vibrations per second), t is the plate thickness (in millimeters), d is the diameter of the plate, E is the elasticity of the plate material, and ρ is the density of the plate material. For fixed values of d , E , and ρ , the graph of the equation is a line, as shown in the figure.



103. Estimate the frequency when the plate thickness is 2 millimeters.
104. Estimate the plate thickness when the frequency is 600 vibrations per second.
105. Approximate the interval for the plate thickness when the frequency is between 200 and 400 vibrations per second.

106. Approximate the interval for the frequency when the plate thickness is less than 3 millimeters.

Conclusions

True or False? In Exercises 107 and 108, determine whether the statement is true or false. Justify your answer.

107. If $-10 \leq x \leq 8$, then $-10 \geq -x$ and $-x \geq -8$.
108. The solution set of the inequality $\frac{3}{2}x^2 + 3x + 6 \geq 0$ is the entire set of real numbers.

Think About It In Exercises 109 and 110, consider the polynomial $(x - a)(x - b)$ and the real number line (see figure).



109. Identify the points on the line where the polynomial is zero.
110. In each of the three subintervals of the line, write the sign of each factor and the sign of the product. For which x -values does the polynomial possibly change signs?
111. **Proof** The arithmetic mean of a and b is given by $(a + b)/2$. Order the statements of the proof to show that if $a < b$, then $a < (a + b)/2 < b$.

$$(i) \ a < \frac{a + b}{2} < b$$

$$(ii) \ 2a < 2b$$

$$(iii) \ 2a < a + b < 2b$$

$$(iv) \ a < b$$

112. **Think About It** Without performing any calculations, match the inequality with its solution. Explain your reasoning.

$$(a) \ 2x \leq -6 \qquad (i) \ x \leq -8 \text{ or } x \geq 4$$

$$(b) \ -2x \leq 6 \qquad (ii) \ x \geq -3$$

$$(c) \ |x + 2| \leq 6 \qquad (iii) \ -8 \leq x \leq 4$$

$$(d) \ |x + 2| \geq 6 \qquad (iv) \ x \leq -3$$