Extension

Applications of Matrices and Determinants

The vertices $(0, 0)$, $(a, 0)$, $(0, b)$, and (a, b) of the square shown in Figure 1.1 can be represented by the column matrices

$$\begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} a \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ b \end{bmatrix}, \text{ and } \begin{bmatrix} a \\ b \end{bmatrix}.$$

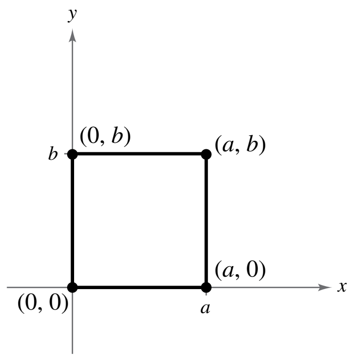


Figure 1.1

To find the image of a square after a transformation, you multiply each vertex by one of the matrices below.

Matrices for Transformation

Reflection in the y-axis

$$\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$$

Reflection in the x-axis

$$\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

Horizontal stretch $(k > 1)$ or shrink $(0 < k < 1)$

$$\begin{bmatrix} k & 0 \\ 0 & 1 \end{bmatrix}$$

Vertical stretch $(k > 1)$ or shrink $(0 < k < 1)$

$$\begin{bmatrix} 1 & 0 \\ 0 & k \end{bmatrix}$$

EXAMPLE 1 Transforming a Square

Find the image of the square with the vertices $(0, 0)$, $(2, 0)$, $(0, 2)$, and $(2, 2)$ after a reflection in the y-axis. Then sketch the square and its image.

Solution

Write the vertices $(0, 0)$, $(2, 0)$, $(0, 2)$, and $(2, 2)$ as the column matrices

$$\begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \end{bmatrix}, \text{ and } \begin{bmatrix} 2 \\ 2 \end{bmatrix}.$$

Then multiply each vertex by the transformation matrix

$$\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}.$$

$$\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \qquad \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 2 \\ 0 \end{bmatrix} = \begin{bmatrix} -2 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 0 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 \\ 2 \end{bmatrix} \qquad \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 2 \\ 2 \end{bmatrix} = \begin{bmatrix} -2 \\ 2 \end{bmatrix}$$

So, the vertices of the image are $(0, 0)$, $(-2, 0)$, $(0, 2)$, and $(-2, 2)$. A sketch of the square and its image is shown in Figure 1.2.

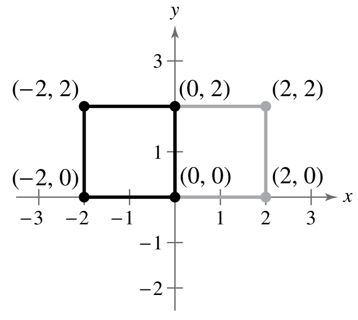


Figure 1.2

You can find the area of a parallelogram using the determinant of a 2×2 matrix.

Area of a Parallelogram

The area of a parallelogram with vertices $(0, 0)$, (a, b), (c, d), and $(a + c, b + d)$ is

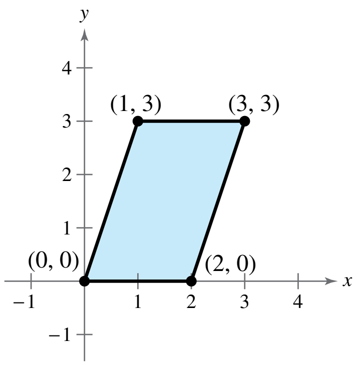
$$\text{Area} = |\det(A)|$$ $|\det(A)|$ is the absolute value of the determinant.

where

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}.$$

EXAMPLE 2 Finding the Area of a Parallelogram

Find the area of the parallelogram whose vertices are $(0, 0)$, $(2, 0)$, $(1, 3)$, and $(3, 3)$, as shown in the figure.



Solution

Let $(a, b) = (2, 0)$, $(c, d) = (1, 3)$, and $(a + c, b + d) = (3, 3)$. Then, evaluate the determinant.

$$\det(A) = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = \begin{vmatrix} 2 & 0 \\ 1 & 3 \end{vmatrix} = (2)(3) - (1)(0) = 6$$

The absolute value of the determinant is the area of the parallelogram.

$$\text{Area} = |\det(A)| = |6| = 6 \text{ square units}$$

Exercises

Transforming a Square **In Exercises 1–4, find the image of the square with the given vertices after the given transformation. Then sketch the square and its image.**

1. $(0, 0)$, $(0, 3)$, $(3, 0)$, $(3, 3)$; horizontal stretch, $k = 2$

2. $(1, 2)$, $(3, 2)$, $(1, 4)$, $(3, 4)$; reflection in the x-axis

3. $(4, 3)$, $(5, 3)$, $(4, 4)$, $(5, 4)$; reflection in the y-axis

4. $(1, 1)$, $(3, 2)$, $(0, 3)$, $(2, 4)$; vertical shrink, $k = \frac{1}{2}$

Finding the Area of a Parallelogram **In Exercises 5–8, find the area of the parallelogram with the given vertices.**

5. $(0, 0)$, $(1, 0)$, $(2, 2)$, $(3, 2)$

6. $(0, 0)$, $(3, 0)$, $(4, 1)$, $(7, 1)$

7. $(0, 0)$, $(-2, 0)$, $(1, 5)$, $(3, 5)$

8. $(0, 8)$, $(8, 2)$, $(0, 0)$, $(8, -6)$