Extension

Operations of Complex Numbers in the Complex Plane

Previously, you learned how to add and subtract vectors geometrically in a coordinate plane. In a similar way, you can add and subtract complex numbers in a complex plane. The complex number $a + bi$ can be represented by the vector $\mathbf{u} = \langle a, b \rangle$. For example, the complex number

$1 + 2i$ is represented by the vector $\mathbf{u} = \langle 1, 2 \rangle$.

To add two complex numbers geometrically, first represent them as vectors **u** and **v**. Then position the vectors (without changing the lengths or directions) so that the initial point of the second vector **v** coincides with the terminal point of the first vector **u**. The sum

$$\mathbf{u} + \mathbf{v}$$

is the vector formed by joining the initial point of the first vector **u** with the terminal point of the second vector **v**, as shown in Figure 1.1.

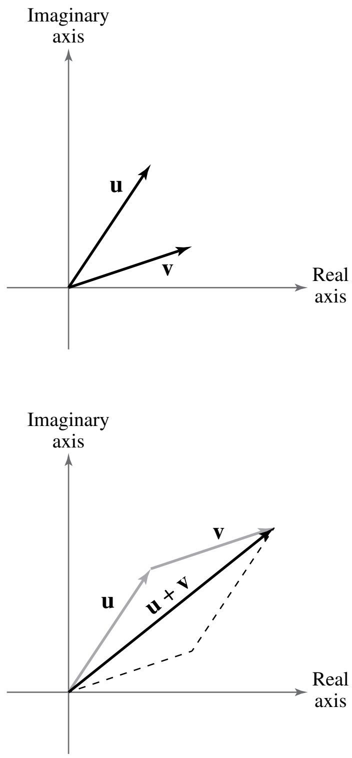


Figure 1.1

EXAMPLE 1 Adding Complex Numbers in the Complex Plane

Find $(1 + 3i) + (2 + i)$.

Solution

Let the complex numbers $1 + 3i$ and $2 + i$ be represented by the vectors $\mathbf{u} = \langle 1, 3 \rangle$ and $\mathbf{v} = \langle 2, 1 \rangle$, respectively. Graph the vector **u** in the complex plane. Then, without changing the length or direction, graph the vector **v** at the terminal point of **u**. Draw $\mathbf{u} + \mathbf{v}$ by joining the initial point of **u** with the terminal point of **v**, see Figure 1.2. So, $\mathbf{u} + \mathbf{v} = \langle 3, 4 \rangle$ and

$$(1 + 3i) + (2 + i) = 3 + 4i.$$

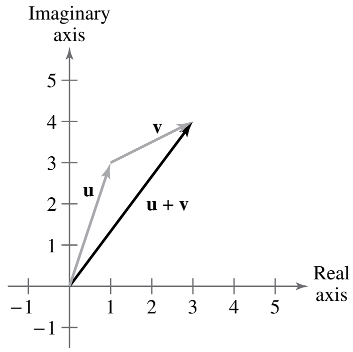
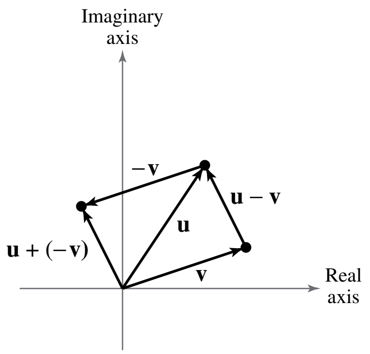


Figure 1.2

To subtract two complex numbers geometrically, you can use directed line segments with the *same* initial point. The difference

$$\mathbf{u} - \mathbf{v}$$

is the vector from the terminal point of **v** to the terminal point of **u**, which is equal to $\mathbf{u} + (-\mathbf{v})$, as shown in the figure below.



EXAMPLE 2 Subtracting Complex Numbers in the Complex Plane

Find $(4 + 2i) - (3 - i)$.

Solution

Let the complex numbers $4 + 2i$ and $3 - i$ be represented by the vectors $\mathbf{u} = \langle 4, 2\rangle$ and $\mathbf{v} = \langle 3, -1\rangle$, respectively. Graph the vector $\mathbf{u}$ in the complex plane. Then graph the vector $-\mathbf{v}$ at the terminal point of $\mathbf{u}$. Draw $\mathbf{u} + (-\mathbf{v})$ by joining the initial point of $\mathbf{u}$ with the terminal point of $-\mathbf{v}$, as shown in Figure 1.3. So, $\mathbf{u} - \mathbf{v} = \mathbf{u} + (-\mathbf{v}) = \langle 1, 3\rangle$ and $(4 + 2i) - (3 - i) = 1 + 3i$.

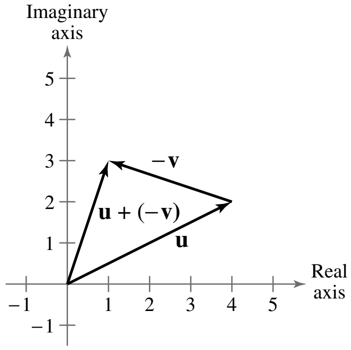
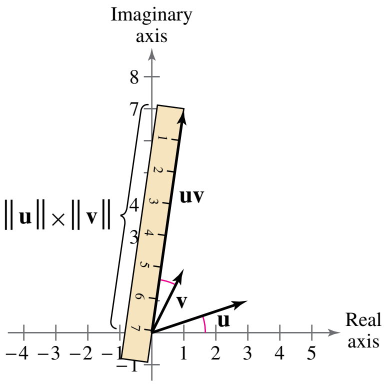


Figure 1.3

The rule for multiplying complex numbers in trigonometric form is

$$z_1z_2 = r_1r_2[\cos(\theta_1 + \theta_2) + i\sin(\theta_1 + \theta_2)].$$

That is, multiply the moduli and add the arguments. The steps for multiplying two complex numbers geometrically using vectors are similar.



1. Represent the complex numbers as vectors $\mathbf{u}$ and $\mathbf{v}$.
2. Multiply $\|\mathbf{u}\|$ and $\|\mathbf{v}\|$ to find the length of the product vector.
3. Add the direction angles of $\mathbf{u}$ and $\mathbf{v}$ to find the direction angle of the product vector.
4. Write the product as a vector and as a complex number.

EXAMPLE 3 Multiplying Complex Numbers in the Complex Plane

Find $(\sqrt{3} + i)(1 + \sqrt{3}i)$.

Solution

Let $\mathbf{u} = \langle \sqrt{3}, 1\rangle$ and $\mathbf{v} = \langle 1, \sqrt{3}\rangle$. Then $\|\mathbf{u}\| = \sqrt{(\sqrt{3})^2 + 1^2} = \sqrt{4} = 2$ and $\|\mathbf{v}\| = \sqrt{1^2 + (\sqrt{3})^2} = \sqrt{4} = 2$. So, the magnitude of the product vector is

$$\|\mathbf{u}\| \times \|\mathbf{v}\| = 2 \times 2 = 4.$$

Find the direction angles of $\mathbf{u}$ and $\mathbf{v}$.

$$\theta_{\mathbf{u}} = \arctan\left(\frac{1}{\sqrt{3}}\right) \qquad \theta_{\mathbf{v}} = \arctan\left(\frac{\sqrt{3}}{1}\right)$$

$$= 30^\circ \qquad\qquad = 60^\circ$$

The sum of the direction angles is $30^\circ + 60^\circ = 90^\circ$. So, the product vector lies on the imaginary axis and is represented in vector form as $\langle 0, 4\rangle$ as shown in Figure 1.4. The complex number form is $0 + 4i$, or $4i$.

So, $(\sqrt{3} + i)(1 + \sqrt{3}i) = 4i$.

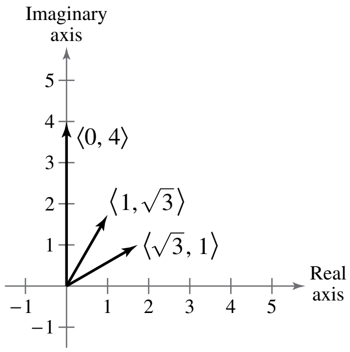
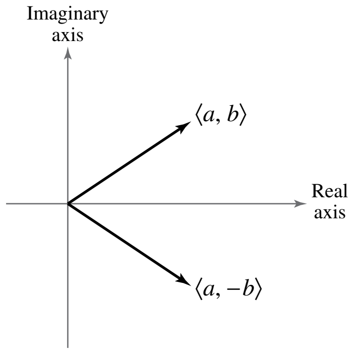


Figure 1.4

You know that the complex conjugate of $a + bi$ is $a - bi$. Written in vector form and graphed in the complex plane, you can see that the complex conjugate is a reflection in the x-axis of the complex number, as shown in the figure.



Let θ be the direction angle for $\langle a, b\rangle$. Then $-\theta$ is the direction angle for $\langle a, -b\rangle$. When multiplying two complex numbers, the direction angles are added. For complex conjugates, this sum is $0°$. So, the product vector lies on the real axis and the product is a real number.

EXAMPLE 4 Finding Complex Conjugates in the Complex Plane

Graph the complex number

$$-3 + i$$

and its conjugate. Write the conjugate as a complex number.

Solution

The complex conjugate is obtained by a reflection of the complex number in the real axis, see Figure 1.5. So, the complex conjugate is $-3 - i$.

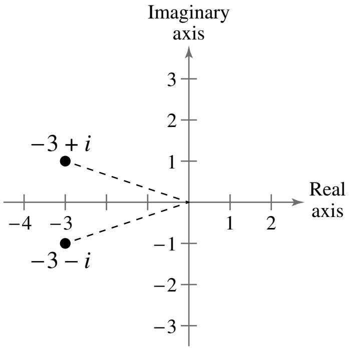


Figure 1.5

Exercises

Adding Complex Numbers in the Complex Plane **In Exercises 1 and 2, find the sum.**

1. $(3 + i) + (2 + 5i)$ **2.** $(8 - 2i) + (2 + 6i)$

Subtracting Complex Numbers in the Complex Plane **In Exercises 3 and 4, find the difference.**

3. $(4 + 2i) - (6 + 4i)$ **4.** $3i - (-3 + 7i)$

Multiplying Complex Numbers in the Complex Plane **In Exercises 5 and 6, find the product.**

5. $(2 + i)(2 - i)$ **4.** $\left(2\sqrt{2} + 2\sqrt{2}i\right)(-4i)$

Finding Complex Conjugates in the Complex Plane **In Exercises 7 and 8, graph the complex number and its conjugate. Write the conjugate as a complex number.**

7. $2 + 3i$ **8.** $5 - 4i$