Extension

Using Matrices to Transform Vectors

Another way to add, subtract, and perform scalar multiplication with vectors is to use matrices. To write a vector as a matrix, use a column matrix. For example, write the vector $\langle 1, 7\rangle$ as the matrix

$$\begin{bmatrix} 1 \\ 7 \end{bmatrix}.$$

EXAMPLE 1 Vector Operations

Let $\mathbf{v} = \langle 2, 4\rangle$ and $\mathbf{w} = \langle 6, 2\rangle$. Use matrices to find each of the following vectors.

a. $3\mathbf{v}$ **b.** $\mathbf{v} + \mathbf{w}$ **c.** $\mathbf{w} - 2\mathbf{v}$

Solution

a. Begin by writing $\mathbf{v} = \langle 2, 4\rangle$ as the matrix $\begin{bmatrix} 2 \\ 4 \end{bmatrix}$. Then

$$\begin{aligned} 3\mathbf{v} &= 3\begin{bmatrix} 2 \\ 4 \end{bmatrix} \\ &= \begin{bmatrix} 3(2) \\ 3(4) \end{bmatrix} \\ &= \begin{bmatrix} 6 \\ 12 \end{bmatrix}. \end{aligned}$$

So, $3\mathbf{v} = \langle 6, 12\rangle$. A sketch of the vector is shown in Figure 1.1.

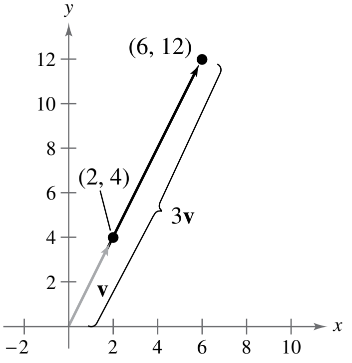

Figure 1.1

b. The sum of $\mathbf{v}$ and $\mathbf{w}$ is

$$\begin{aligned} \mathbf{v} + \mathbf{w} &= \begin{bmatrix} 2 \\ 4 \end{bmatrix} + \begin{bmatrix} 6 \\ 2 \end{bmatrix} \\ &= \begin{bmatrix} 2 + 6 \\ 4 + 2 \end{bmatrix} \\ &= \begin{bmatrix} 8 \\ 6 \end{bmatrix}. \end{aligned}$$

So, $\mathbf{v} + \mathbf{w} = \langle 8, 6\rangle$. A sketch of the vector is shown in Figure 1.2.

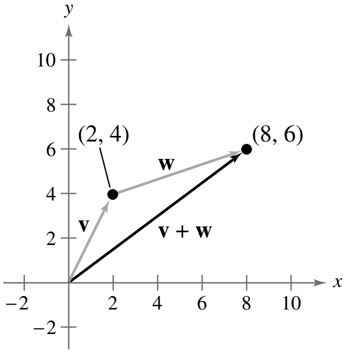

Figure 1.2

c. The difference of $\mathbf{w}$ and $2\mathbf{v}$ is

$$\begin{aligned} \mathbf{w} - 2\mathbf{v} &= \begin{bmatrix} 6 \\ 2 \end{bmatrix} - 2\begin{bmatrix} 2 \\ 4 \end{bmatrix} \\ &= \begin{bmatrix} 6 \\ 2 \end{bmatrix} - \begin{bmatrix} 4 \\ 8 \end{bmatrix} \\ &= \begin{bmatrix} 6 - 4 \\ 2 - 8 \end{bmatrix} \\ &= \begin{bmatrix} 2 \\ -6 \end{bmatrix}. \end{aligned}$$

So, $\mathbf{w} - 2\mathbf{v} = \langle 2, -6\rangle$. A sketch of the vector is shown in Figure 1.3. Note that the figure shows the vector difference $\mathbf{w} - 2\mathbf{v}$ as $\mathbf{w} + (-2\mathbf{v})$.

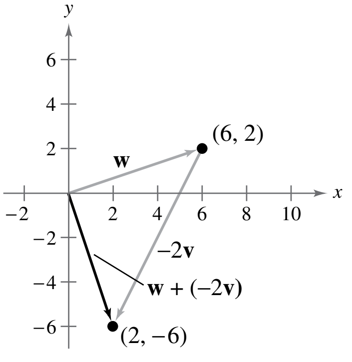

Figure 1.3

One way to transform a vector in the coordinate plane is to multiply the vector by a square matrix. To transform a vector using matrix multiplication, two conditions must be met.

1. The number of columns in the transformation matrix A must equal the number of rows in the vector column matrix $\mathbf{v}$.
2. The product $A\mathbf{v}$ of the transformation matrix and the vector column matrix must have the same dimensions as the vector column matrix $\mathbf{v}$.

$$\begin{array}{ccccc} & & \text{Same dimensions} & & \\ A & \times & \mathbf{v} & = & A\mathbf{v} \\ n \times n & & n \times 1 & & n \times 1 \\ & \text{Equal} & & & \end{array}$$

The product of a transformation matrix and a vector is another vector that is a transformation of the original vector.

EXAMPLE 2 Vector Transformations

Find the product $A\mathbf{v}$, where $A = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ and $\mathbf{v} = \langle 1, 3 \rangle$, and describe the transformation.

Solution

You can find $A\mathbf{v}$ because the number of columns in the transformation matrix is equal to the number of rows in the vector column matrix, and the product of the transformation matrix and the vector column matrix has the same dimensions as the vector column matrix.

$$\begin{aligned} A\mathbf{v} &= \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} 1 \\ 3 \end{bmatrix} \\ &= \begin{bmatrix} 1(1) + 0(3) \\ 0(1) + (-1)(3) \end{bmatrix} \\ &= \begin{bmatrix} 1 \\ -3 \end{bmatrix} \end{aligned}$$

So, $A\mathbf{v} = \langle 1, -3 \rangle$. A sketch of the vectors is shown in the Figure 1.4. Notice that the product of A and $\mathbf{v}$ results in a vector that is reflected in the x-axis.

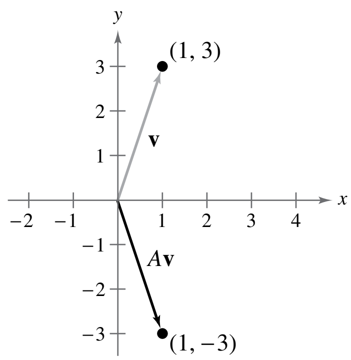

Figure 1.4

Exercises

Vector Operations **In Exercises 1 and 2, use matrices to find (a) $4\mathbf{v}$, (b) $\mathbf{v} + \mathbf{w}$, (c) $\mathbf{w} - \mathbf{v}$, and (d) $2\mathbf{v} + 3\mathbf{w}$. Then sketch the specified vector operations geometrically.**

1. $\mathbf{v} = \langle 0, 4 \rangle$, $\mathbf{w} = \langle 3, 2 \rangle$

2. $\mathbf{v} = \langle 1, 5 \rangle$, $\mathbf{w} = \langle 2, 0 \rangle$

Vector Transformations **In Exercises 3 and 4, find $A\mathbf{v}$. Then sketch $\mathbf{v}$ and $A\mathbf{v}$ and describe the transformation.**

3. $A = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$, $\mathbf{v} = \langle 7, 0 \rangle$

4. $A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$, $\mathbf{v} = \langle 2, 3 \rangle$

Writing **In Exercises 5 and 6, let $\mathbf{v} = \langle a, b \rangle$. Explain why $A\mathbf{v}$ cannot be found.**

5. $A = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

6. $A = \begin{bmatrix} 1 & 0 \\ 0 & -1 \\ 1 & 0 \end{bmatrix}$